PILOT Final Review

Differential Equations

Johns Hopkins University

Fall 2024

As you prepare for the final examination, please consider the following resources:

- PILOT webpage for ODEs:
  `https://jhu-ode-pilot.github.io/FA24/`
  - Check on the weekly problem sets.
  - Find the review problem set for this semester.
- *Homework sets*, *quizzes*, and *practice final set* provided by the instructor.
- PILOT Review Session for `AS.110.302` Differential Equations and Application. (You are here.)
  1. We will first go over all contents for this semester.
  2. In the end, we will open the poll to you. Please indicate which problems from the PSets or Review Set that you want us to go over.

Contents:

This slide deck or its printable PDF version can be found on the PILOT webpage.

Containing Extra Material

Please keep note that some sections and chapters are not covered by the instructor this semester.

- The slide deck records the contents for `AS.110.302` Differential Equations and Applications at *Johns Hopkins University*.
- The notes summaries some contents, theorems, and formulas from the course as well as the textbook that is used:
  - *Elementary Differential Equations and Boundary Value Problems* by William E. Boyce, Richard C. Diprima, and Douglas B. Meade.
- The notes has been compiled and modified every term since Fall 2022 for PILOT by James Guo. It might contain minor typos or errors. Please point out any notable error(s).

Best regards, James Guo.
December 2024.

Part 1:
Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

Differential equations can be classified by their properties:

- Ordinary Differential Equations (ODEs) involves ordinary derivatives ($\frac{dy}{dt}$), while Partial Differential Equations (PDEs) involves partial derivatives ($\frac{\partial y}{\partial t}$).
- Single equation involves one unknown and one equation, while System of equations involves multiple unknowns and multiple equations.
- The order of the differential equation is the order of the highest derivatives term.
- Linear differential equations has only linear dependent on the function, while non-linear differential equations has non-linear dependent on the function.

ODEs can be used for modeling. During modeling, it follows the following steps:

1. Construction of the Models,
2. Analysis of the Models,
3. Comparison of the Models with Reality.

The physics model for half life indicates the relationship between half life (τ) of a substance of amount $N(t)$ with initial amount N_0 at a time t is:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{\tau}},$$

where the rate of decay (λ) and half life (τ) are related by:

$$\tau \times \lambda = \log 2.$$

For ODEs in form $\dfrac{dy}{dt} + a(t)y = b(t)$, the integrating factor is:

$$\mu(t) = \exp\left(\int a(t)dt\right).$$

For ODEs in form $M(t) + N(y)\dfrac{dy}{dt} = 0$, it can be separated by:

$$M(t)dt + N(y)dy = 0.$$

The existence and uniqueness for Initial Value Problem (IVP) depend on cases:

- For an IVP in simple form:
$$\begin{cases} \dfrac{dy}{dt} = a(t) + b(t), \\ y(t_0) = y_0. \end{cases}$$
If $a(t)$ and $b(t)$ are continuous on an interval $[\alpha, \beta]$ and $t_0 \in [\alpha, \beta]$. Then, there exists a uniqueness solution y for $[\alpha, \beta]$ to the IVP.

- For an IVP in general form:

$$\begin{cases} \dfrac{dy}{dt} = f(t,y), \\ y(t_0) = y_0. \end{cases}$$

For $t_0 \in I = [a,b]$, $y_0 \in J = [c,d]$, if $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous on interval $I \times J = [a,b] \times [c,d]$. Then, there exists a unique solution on a smaller interval $I' \times J' \subset I \times J$.

Autonomous ODEs are in form of:

$$\frac{dy}{dt} = f(y).$$

The stability (stable/semi-stable/unstable) of equilibrium can be determined by phase lines, *i.e.*, the zeros of the function $f(t)$.

The logistic population growth model with population (y), growing rate (r), and carrying capacity (k) is given by:

$$\begin{cases} \dfrac{dy}{dt} = r\left(1 - \dfrac{y}{k}\right)y, \\ y(0) = y_0. \end{cases}$$

The solution for Logistic Population Growth is:

$$y(t) = \frac{ky_0}{(k - y_0)e^{-rt} + y_0}.$$

The condition for a function in form $M(x,y)+N(x,y)\dfrac{dy}{dx}=0$ to be exact is:

$$\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}.$$

For solving Exact ODEs, either finding $\int M(x,y)dx+h(y)$ or $\int N(x,y)dy+h(x)$ and taking partials again to fit gives the solution $\Psi(x,y)=C$.

For not exact cases, the integrating factor is:

$$\mu(t)=\exp\left(\int\frac{M_y-N_x}{N}dx\right)\ \text{ or }\ \mu(t)=\exp\left(\int\frac{N_x-M_y}{M}dy\right).$$

Consider the linear homogeneous ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is:

$$r^2 + pr + q = 0.$$

With solutions r_1 and r_2, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

If the solutions r_1 and r_2 are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t} \cos(\beta t) + c_2 e^{\lambda t} \sin(\beta t).$$

If the solutions r_1 and r_2 are repeated, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

To form a fundamental set of solutions, the solutions need to be linearly independent, in which the Wronskian (W) must be non-zero, meaning that:

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Consider IVP in form:

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_1, y'(t_0) = y_2. \end{cases}$$

The interval I containing t_0 has $p(t)$, $q(t)$, and $g(t)$ continuous on it. Then, there is a unique solution $y(t)$ and twice differentiable on the interval I.

If $y_1(t)$ and $y_2(t)$ are solutions to $l[y] = 0$, then the solution $c_1 y_1(t) + c_2 y_2(t)$ are also solutions for all constants $c_1, c_2 \in \mathbb{R}$.

Consider the equation $y'' + py' + qy = 0$, the Wronskian for the solutions are:

$$W[y_1, y_2] = C \exp\left(-\int p dt\right),$$

where C is independent of t but depends on y_1 and y_2.

For non-linear second order homogeneous ODEs, when one solution $y_1(t)$ is given, the other solution is in form:

$$y_2(t) = u(t) \times y_1(t).$$

Let the differential equation be:

$$Ay''(t) + By'(t) + Cy(t) = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$ and $y_2(t)$ be the two homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

- Undetermined Coefficient: A guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \sin(ax)$
Exp.:	e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: The particular solution is:

$$y_p = y_1(t) \int \frac{-y_2(t) \times g(t)}{W} dt + y_2(t) \int \frac{y_1(t) \times g(t)}{W} dt.$$

For higher order IVP in form:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots\ , y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

If $P_0(t)$, $P_1(t)$, $\cdots$, $P_{n-1}(t)$, and $g(t)$ are continuous on an interval I containing t_0. Then there exists a unique solution for $y(t)$ on I.

The higher order homogeneous ODEs are in form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

By computing the characteristic equation:

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0.$$

With solutions $r_1, r_2, \cdots, r_n$, the general solution is:

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t} + \cdots + c_ne^{r_nt}.$$

Note that the complex solutions can still be converted to sines and cosines, while repeated roots multiply a t on the repeated solutions.

To obtain the fundamental set of solutions, the Wronskian (W) must be non-zero, where Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}.$$

- Alternation to the Wronskian: By definition of linear independence, $f_1, f_2, \cdots, f_n$ are independent on I is equivalent to the expression where $k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$ if and only if $k_i = 0$.

For higher order ODEs in the form of:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots,y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

Its Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = Ce^{\int P_{n-1}(t)dt},$$

where C is independent of t but depend on $y_1, y_2, \cdots, y_n$.

Let the differential equation be:

$$L[y^{(n)}(t), y^{(n-1)}(t), \cdots, y(t)] = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$, $y_2(t)$, $\cdots$, $y_n(t)$ be all homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

- Undetermined Coefficient: Same as in degree 2, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \sin(ax)$
Exp.:	e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: The particular solution is:
$$y_p = y_1(t)\int \frac{W_1 g}{W}dt + y_2(t)\int \frac{W_2 g}{W}dt + \cdots + y_n(t)\int \frac{W_n g}{W}dt,$$
where W_i is defined to be the Wronskian with the i-th column alternated into $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - Ir) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - Ir) \cdot \boldsymbol{\xi} = \mathbf{0}.$$

The solution to the ODE is:

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + c_2 \boldsymbol{\xi}^{(2)} e^{r_2 t} + \cdots + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}.$$

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det\left(\Psi(t)\right).$$

The system is linearly independent if the Wronskian is non-zero.
For the linear system in form:

$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A, which is the sum of the diagonals, that is:

$$W = Ce^{\int \text{trace}\, A dt} = Ce^{\int (A_{1,1}+A_{2,2}+\cdots+A_{n,n})dt}.$$

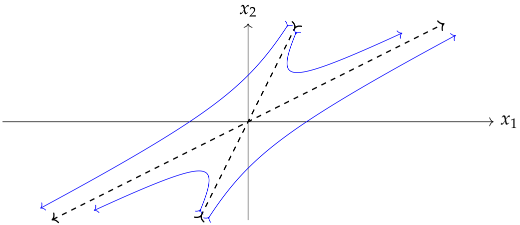
For repeated eigenvalue r with only one eigenvector, if a given a solution is $\mathbf{x}^{(1)} = \boldsymbol{\xi} e^{rt}$, the other solution would be:

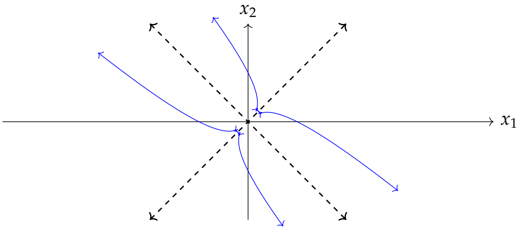
$$\mathbf{x}^{(2)} = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt},$$

where $(A - Ir) \cdot \boldsymbol{\eta} = \boldsymbol{\xi}$.

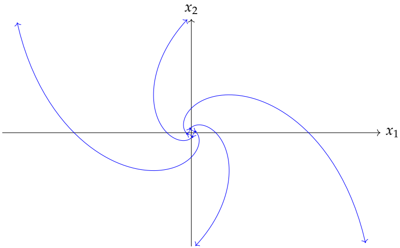
In particular, we can sketch the linear system of $\mathbb{R}^2$ in terms of phase portraits given the eigenvalues and eigenvectors.

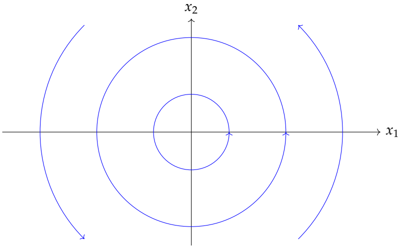
- For a node graph, we have it as (directions might vary):



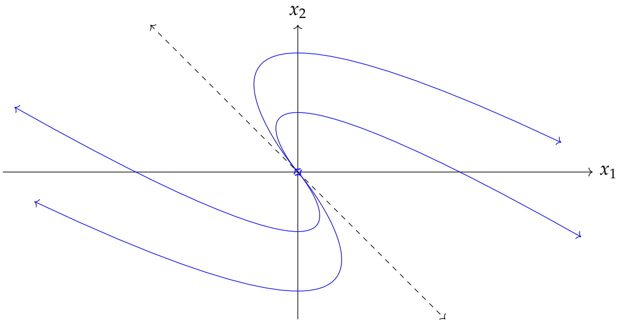


- For a spiral/center graph, we have it as (directions might vary):





- For repeated eigenvalues, the solution depends is (directions might vary):



The exponential of Matrix is defined to be:

$$\exp(tA) = I + \sum_{n=1}^{\infty} \frac{(tA)^n}{n!},$$

where A^n is the result of n square matrices of A multiplying themselves.

The special case of fundamental matrix is defined to be Φ where:

$$\begin{cases} \Phi' = A \cdot \Phi, \\ \Phi(0) = I, \end{cases}$$

so that the fundamental matrix Φ can be calculated by:

$$\Phi(t) = \Psi(t) \cdot \Psi^{-1}(0).$$

Let the differential equation be:

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{g}(t),$$

where $\mathbf{g}(t)$ is a smooth vector-valued function. Let ϕ be its fundamental matrix, then the non-homogeneous cases can be solved by the following approaches:

- Diagonalization: Diagonalization utilizes T as the matrix of eigenvectors and D as the diagonal matrix of eigenvalues. Accordingly, let $\mathbf{x} = T\mathbf{y}$.
  Then, $\mathbf{x}' = T\mathbf{y}' = AT\mathbf{y} + \mathbf{g} = TD\mathbf{y} + \mathbf{g}$, which means that $\mathbf{y}' = D\mathbf{y} + T^{-1}\mathbf{g}$ and the differential equation is now degenerated.

- Undetermined Coefficient: Same as in single equations, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $\mathbf{g}(t)$. Some brief strategies are:

Non-homogeneous Comp. in $\mathbf{g}(t)$	Guess
Polynomials: $\sum_{i=0}^{d} \mathbf{a_i} t^i$	$\sum_{i=0}^{d} \mathbf{c_i} t^i$
Trig.: $\mathbf{a_1} \sin(b_1 t)$ and $\mathbf{a_2} \cos(b_2 t)$	$\mathbf{c_1} \sin(b_1 x) + \mathbf{c_2} \sin(b_2 x)$
Exp.: $\mathbf{a} e^{bt}$	$\mathbf{c} e^{bt}$

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: Variation of parameters utilizes that:

$$\Psi \cdot \mathbf{u}' = \mathbf{g},$$

where this equation can be solved by:

$$u_i' = \frac{W_i}{\det(\Psi)},$$

where W_i is defined by the Wronskian of the matrix replacing the i-th column with $\mathbf{g}(t)$.
There, the particular solution is:

$$\mathbf{x}_p = \Psi \cdot \mathbf{u}.$$

For non-linear system $\mathbf{x}' = \begin{pmatrix} F \\ G \end{pmatrix} \mathbf{x}$, if $F, G \in C^2$, i.e. locally linear, the approximation at critical point (x_0, y_0) is:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \mathbf{J}(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where Jacobian is:

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}.$$

When $\mathbf{x} = \begin{pmatrix} F(y) \\ G(x) \end{pmatrix}$, it can be solved implicitly for:

$$\frac{dy}{dx} = \frac{G(x)}{F(y)}.$$

For linearized system with eigenvalues r_1, r_2:

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$: $r_1 < r_2 < 0$ indicates an asymptotically stable node, $r_1 < 0 < r_2$ indicates a mostly unstable saddle, and $0 < r_1 < r_2$ indicates an unstable node. Note that these will not change for the non-linear case.
2. If $r_1 = r_2$: $r_1 = r_2 < 0$ indicates a asymptotically stable node and $r_1 = r_2 > 0$ indicates an unstable node. The stability preserves but the shape either node or spiral.
3. If $r_1, r_2 \in \mathbb{C}$ and $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) \neq 0$: $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) > 0$ indicates an unstable spiral and $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) < 0$ indicates an asymptotically stable spiral. Note that these will not change for the non-linear case.
4. If $r_1, r_2 \in \mathbb{C}$ and $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) = 0$: That indicates a stable center. In the non-linear case, the shape is either spiral or center, but the stability is in-determinant.

The stability can be concluded as follows:

Eigenvalues	Linear System		Nonlinear System	
	Type	Stability	Type	Stability
Eigenvalues are λ_1 and λ_2				
$0 < \lambda_1 < \lambda_2$	Node	Unstable	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable	Node or Spiral Point	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable	Node or Spiral Points	Asymptotically Stable

Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$				
$\alpha > 0$	Spiral Point	Unstable	Spiral Point	Unstable
$\alpha = 0$	Center	Stable	Center or Spiral Point	Indeterminate
$\alpha < 0$	Spiral Point	Asymptotically Stable	Spiral Point	Asymptotically Stable

A closed trajectory or periodic solution repeats back to itself with period τ:

$$\begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Closed trajectories with either side converging to/diverging from the solution is a limit cycle.

A Cartesian coordinate can be converted by:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ rr' = xx' + yy', \\ r^2\theta' = xy' - yx'. \end{cases}$$

For a linear system $x = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$ with $F, G \in C^1$:

1. A closed trajectory of the system must enclose at least 1 critical point.
2. If it only encloses 1 critical point, then that critical point cannot be saddle point.
3. If there are no critical points, there are no closed trajectories.
4. If the unique critical point is saddle, there are no trajectories.
5. For a simple connected domain D in the xy-plane with no holes. If $F_x + G_y$ had the same sign throughout D, then there is no closed trajectories in D.

The Laplace Transformation of a function f is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Note that Laplace Transformation can be used on non-continuous functions by utilizing step functions.

Laplace Transformation has the following properties:

1. Laplace Transformation is a linear operator:
$$\mathcal{L}\{f+\lambda g\} = \mathcal{L}\{f\} + \lambda\mathcal{L}\{g\}$$
2. Laplace Transformation for derivatives:
$$\begin{aligned}\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0),\\ \mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0),\\ &\vdots\\ \mathcal{L}\{f^{(n)}(t)\} &= s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0).\end{aligned}$$
3. First Shifting Theorem:
$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c).$$

The Laplace Transformations can be used for solving IVP, where the inverse helps to find the original function prior to transformation.

The Laplace Transformations for elementary functions are given in the following table, note that they can still be calculated by its definition:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$t^n, n \in \mathbb{Z}_{>0}$	$\frac{n!}{s^{n+1}}, s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}, s > 0$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$

The step functions are defined by:

$$u_c(t) = u(t-c) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

And the Laplace Transformations of the step function is:

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}.$$

The step function forms the Second Shifting Theorem:

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s).$$

The idealized unit impulse function $\delta(t)$, or *Dirac delta function*, satisfies the properties that:

$$\delta(t) = 0 \text{ for } t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

There is no ordinary function satisfying the idealized unit impulse function, so it is a generalized function.
A unit impulse at an arbitrary point $t = t_0$, denoted by $\delta(t - t_0)$, follows that:

$$\delta(t) = 0 \text{ for } t \neq t_0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t - t_0)dt = 1.$$

The Laplace Transformation of the impulse function is:

$$\mathcal{L}\{\delta(t - c)\} = e^{-cs}.$$

The convolution of f and g, denoted $(f * g)$, is defined as:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau.$$

The convolution $f * g$ has many of the properties of ordinary multiplication:

1. Commutativity: $f * g = g * f$;
2. Distributivity: $f * (g + h) = f * g + f * h$;
3. Associativity: $(f * g) * h = f * (g * h)$;
4. Zero Property: $f * 0 = 0 * f = 0$, where 0 is a function that maps any input to 0.

The Laplace Transformation of the convolution of f and g is:

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s).$$

A power series is an infinite series in the form:

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots ,$$

where a_n is the coefficient for term n and c is the center of the approximation.

A power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ *converge* at a point x if:

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n (x-x_0)^n \text{ exists for that } x.$$

A power series *converges pointwise* on X if it converges on every $x \in X$.

A power series converges absolutely at a point x if the power series:

$$\sum_{n=0}^{\infty} |a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||x-x_0|^n \text{ converges.}$$

Note that absolute converges implies convergence, but the converse is not true.

Here are some properties of series:

1. (Ratio test). If $a_n \neq 0$, and if for a fixed value of x, and:
$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{x_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L,$$
then the power series converges absolutely at x if $|x-x_0|L < 1$ and diverges if $|x-x_0|L > 1$.
2. (Monotonic property). If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at $x = x_1$, then it converges absolutely for $|x-x_0| < |x_1-x_0|$. If it diverges at $x = x_1$, then it diverges for $|x-x_0| > |x_1-x_0|$.
3. (Radius of convergence). Let $\rho > 0$ be such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$, then ρ is the *radius of convergence* and $(x_0-\rho, x_0+\rho)$ is the *interval of convergence*.

Also, we note that power series can be added or subtracted term-wise. They can also be multiplied and divided by having divisions of terms.

Recall that by Taylor theorem, suppose $f \in C^{\infty}$, then we can form the Taylor polynomial as a power series, with coefficient:

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

In particular, if f has a Taylor polynomial at x_0 with a positive radius of convergence, we say the series is *analytic* at x_0.

Here, we are thinking of the second order homogeneous differential equation, namely:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0.$$

Additionally, we suppose that P, Q, and R are polynomials and have no factor common factor $(x - c)$. Thus, we have $P(x_0) \neq 0$ being an ordinary point. When $P(x_0) = 0$, it is a singular point (or pole).

When we generalize, we will have:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

where p and q are any functions. Similarly, consider x_0 where both p and q are analytic, x_0 is *ordinary*, otherwise, it is *singular*. Here, we say $p(x)$ has singularity of a pole at x_0 of order n if:

$$(x - x_0)^n p(x) \text{ is analytic at } x_0.$$

Assuming absolute convergence, one can apply the derivative operator on the sequence, that is:

$$\begin{aligned}\frac{d}{dx}\left[\lim_{N\to\infty}\sum_{n=0}^{N} a_n(x-x_0)^n\right] &= \lim_{N\to\infty}\left[\frac{d}{dx}\sum_{n=0}^{N} a_n(x-x_0)^n\right] \\ &= \lim_{N\to\infty}\sum_{n=1}^{N} a_n n(x-x_0)^{n-1}.\end{aligned}$$

Often, we we apply the derivative operator, we will notice some *recurrence relation*, that is the successive coefficients can be evaluated one by one.
In particular, when we have a power series:

$$\varphi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

by taking the m-th derivative and evaluating it at 0, we will have:

$$\frac{d^m \varphi}{dx^m}(x_0) = m! a_m.$$

In the section, we go back to the focus of:

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

where P, Q, R are polynomials with no common factors.
For the Euler's equation, we consider the differential equation in the form:

$$x^2y'' + \alpha xy' + \beta y = 0.$$

Then, $|x|^r$ is a solution to the above differential equation if r is a solution to $r(r-1) + \alpha r + \beta = 0$.

Let r_1, r_2 be the roots of $r(r-1) + \alpha r + \beta = 0$, then the solution to the differential equation can be represented by:

- When $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then:
$$y(x) = c_1|x|^{r_1} + c_2|x|^{r_2}.$$
- When $r_1, r_2 \in \mathbb{R}$ and $r := r_1 = r_2$, then:
$$y(x) = c_1|x|^r + c_2 \log|x| \cdot |x|^r.$$
- When $r_1, r_2 = \lambda + i\mu \in \mathbb{C}$ and $\mu \neq 0$, then:
$$y(x) = c_1|x|^\lambda \cos\left(\mu \log|x|\right) + c_2|x|^\lambda \sin\left(\mu \log|x|\right).$$

Now, we want to research on the case when x_0 is a regular singular point, that is for equation:

$$y'' + p(x)y' + q(x)y = 0,$$

and x_0 satisfies that:

1. x_0 is a singular point, and
2. $p(x)$ has a pole of order 1 and $q(x)$ has a pole of order no more than 2.

A singular point that is not regular is a irregular singular point.

Without loss of generality, we may horizontally shift the equation to obtain that $x = 0$ is a regular singular point. Then, we may write:

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \qquad \text{and} \qquad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$$

on some interval $|x| < \rho$ within the radius of convergence.

Hence, we may multiply x^2 on both side, giving us that:

$$x^2y'' + x\underbrace{\left(xp(x)\right)}_{p(\tilde{x})}y' + \underbrace{\left(x^2q(x)\right)}_{\tilde{q}(x)}y = 0,$$

in which $\tilde{p}$ and $\tilde{q}$ are analytic at $x = 0$. Then, we will be able to Euler Equations to solve for the differential equation with respect to $\tilde{p}$ and $\tilde{q}$.

The numerical approximation focuses on first-order initial value problem:

$$\begin{cases} \dfrac{dy}{dt} = f(t, y), \\ y(t_0) = y_0. \end{cases}$$

By the *Existence and Uniqueness Theorem*, a unique solution exists for some rectangular region containing (t_0, y_0) when f and $\frac{\partial f}{\partial y}$ are continuous. With this foundation, we may apply Euler's method on such region. (*Note that* out of the region, the approximation would not be accurate.)

Euler's method recursively applies the following function:

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n), \quad n = 0, 1, 2, \cdots,$$

and when the steps are constrained to be a constant h, we have:

$$y_{n+1} = y_n + hf(t_n, y_n), \quad n = 0, 1, 2, \cdots.$$

Typically, Euler's method incurs error, whereas some typical issues are:

1. When the step size h is too big, the error is significant.
2. When the step size h is too small, the cost of calculation is expensive.
3. The computation does address the asymptotic behaviors.
4. When the vector field has steep components, the approximation differs more.

Euler's method can be analyzed by using the *Fundamental Theorem of Calculus*, that is:

$$\begin{aligned} y(t) &= y(t_n) + \int_{t_n}^{t} f\big(s, y(s)\big)ds \\ &\approx y(t_n) + \sum_{t_0 \le t_i < t_{i+1} \le t} f(t_i, y_n)(t_{i+1} - t_i), \end{aligned}$$

in which we may establish the improved Euler's Method, by:

$$y_{n+1} = y_n + h\left(\frac{f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))}{2}\right),$$

by considering the trapezoid approach for *Riemann sum*. Since the $f(t, y)$ depends only on t and not on y, then solving differential equation reduced from $y' = f(t, y)$ to integrating $f(t)$, which makes the improved Euler's Method into:

$$y_{n+1} = y_n + \frac{h}{2}\big(f(t_n) + f(t_n + h)\big).$$

Part 2: Open Poll

We will work out some sample questions.

- Submit your question with the form:
  `https://forms.office.com/r/qZKvgpQBae?origin=lprLink`

Congratulations for completing PILOT for Differential Equations this semester.

End Remarks

We hope that you have consolidated your knowledge, learned more concepts, and better understood the materials through the semester. We wish the best of luck for your final examination as well as the rest of your academic career.