PILOT

Problem Set 12

Differential Equations

Fall 2024

Up to this week, we are wrapping up the nonlinear system with a special case of limit cycles. As we finish up, please take a moment to review what we have learned together.

- Concepts:
  - Nonlinear System
  - Locally Linear System
  - Jacobian Matrix
- Models:
  - Predator-Pray Model
  - Competing Species Model
  - Limit Cycles

As you are familiar with the above models, we are also moving towards of utilizing series to solve second order linear differential equations.

1. (Limit Cycles). Determine the periodic solution, if there are any, of the following system:
$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \\ y' = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \end{cases}$$

2. (Converging Sequences). In this question, we will review some common power series.
   (a) Construct the power series of e^x, $\sin x$, and $\cos x$ centered at 0.
   (b) Consider the following power series:
   $$\sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!}.$$
   Identify if such series converges. Compute the limit if the series converges.

3. (Analytic Function). Recall the definition of analytic function in class defined for $f : \mathbb{R} \to \mathbb{R}$ at a fixed point $x \in \mathbb{R}$.
   (a) Write down the power series of $f(x) = \frac{1}{1-x}$ around 0. Show that $f(x)$ is analytic at 0.

   Here, we extend the definition to function with complex input and complex output, say $f : \mathbb{C} \to \mathbb{C}$. We can similarly define f being analytic at $x_0 \in \mathbb{C}$ when the function has a positive radius of convergence at x_0.

(b) Convince yourself that if $f : \mathbb{C} \to \mathbb{C}$ is analytical over $\mathbb{C}$ and $f(\mathbb{R}) \subset \mathbb{R}$, then $f\mid_{\mathbb{R}}$ is analytical $\mathbb{R}$.

By Cauchy-Riemann equations, $f(z) := f(x + \mathrm{i}y)$ being analytical is identical with:

$$\frac{\partial f}{\partial x} = -\mathrm{i}\frac{\partial f}{\partial y}.$$

(c) Show that the Möbius transform $\psi : \mathbb{C} \to \mathbb{C}$ such that $\psi_a(z) := \frac{a-z}{1-\bar{a}z}$ is analytic on $\mathbb{C} \setminus \{1/\bar{a}\}$. Conclude with a condition for which $\psi_a(z)\mid_{\mathbb{R}}$ is analytic on some set A.

4. (Recurrence Relation). Solve the following differential equation using power series method. Include the recurrence relation.

$$y'' + y = 0.$$

Tip of the Week

If you ever experience a behavioral health crisis or are concerned about someone else, call and connect with JHU's Behavioral Health Crisis Support Team 24/7 at (410) 516-9355. Find out more information at `https://wellbeing.jhu.edu/bhcst`.