Problem Set 7: Solutions to Additional Questions

Differential Equations

Fall 2024

1. (PDEs: Wave Equation). The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length L, the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x, t)$:

$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x, 0) = \sin\left(\dfrac{2\pi x}{L}\right), & \\ & \dfrac{\partial u}{\partial t}(x, 0) = \sin\left(\dfrac{5\pi x}{L}\right), & \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0, t) = u(L, t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$

where c is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*, $u(x, t) = v(x) \cdot w(t)$, and attempt to obtain a general solution that is *non-trivial*.

Hint: Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n \in \mathbb{Z}^+}$ forms an orthonormal basis (cf. §5.2).

Solution:

With the method of separation, we insert the separations back to the system of equation to obtain:

$$v(x)w''(t) = c^2 v''(x)w(t).$$

Now, we apply the separation and set the common ratio to be λ:

$$\frac{v''(x)}{v(x)} = \frac{1}{c^2} \cdot \frac{w''(t)}{w(t)} = \lambda.$$

Reformatting the boundary condition gives use the following initial value problem:

$$\begin{cases} v''(x) - \lambda v(x) = 0, \\ v(0) = v(L) = 0. \end{cases}$$

As a second order linear ordinary differential equation, we discuss all following cases:

- If $\lambda = 0$, then $v(x) = a + Bx$ and by the initial condition, $A = B = 0$, which gives the trivial solution, *i.e.*, $v(x) = 0$;
- If $\lambda = \mu^2 > 0$, then we have $v(x) = Ae^{-\mu x} + Be^{\mu x}$ and again giving that $A = B = 0$, or the trivial solution;
- Eventually, if $\lambda = -\mu^2 < 0$, then we have the solution as:

$$v(x) = A\sin(\mu x) + B\cos(\mu x),$$

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and the initial conditions gives us that:

$$\begin{cases} v(0) = B = 0, \\ v(L) = A\sin(\mu L) + B\cos(\mu L) = 0, \end{cases}$$

where A is arbitrary, $B = 0$, and $\mu L = m\pi$ positive integer m.
Overall, the only non-trivial solution would be:

$$v_m(x) = A\sin(\mu_m x), \text{ where } \mu_m = \frac{m\pi}{L}.$$

Eventually, by inserting back $\lambda = -\mu_m^2$, we have $\lambda = -m^2\pi^2/L^2$, giving the solution to $w_m(t)$, another second order linear ordinary differential equation, as:

$$w_m(t) = C\cos(\mu_m ct) + D\sin(\mu_m ct), \text{ with } C, D \in \mathbb{R}.$$

By the *principle of superposition*, we can have our solution in the form:

$$u(x,t) = \sum_{m=1}^{\infty} [a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)] \sin(\mu_m x),$$

where our coefficients a_m and b_m have to be chosen to satisfy the initial conditions for $x \in [0, L]$:

$$u(x,0) = \sum_{m=1}^{\infty} a_m \sin(\mu_m x) = \sin\left(\frac{2\pi x}{L}\right),$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} c\mu_m b_m \sin(\mu_m x) = \sin\left(\frac{5\pi x}{L}\right).$$

Since we are hinted that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis, we now know that except for the following:

$$a_2 = 1 \text{ and } c\mu_5 b_5 = 1,$$

all the other coefficients are zero, so we have:

$$u(x,t) = \boxed{\cos\left(\frac{2\pi ct}{L}\right)\sin\left(\frac{2\pi x}{L}\right) + \frac{L}{5\pi c}\sin\left(\frac{5\pi ct}{L}\right)\sin\left(\frac{5\pi x}{L}\right)}.$$

2. (*Putnam 2023*: First Positive Root). Determine the smallest positive real number r such that there exists differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying:

   - $f(0) > 0$,
   - $g(0) = 0$,
   - $|f'(x)| \le |g(x)|$ for all x,
   - $|g'(x)| \le |f(x)|$ for all x, and
   - $f(r) = 0$.

   You may give an answer *without* a rigorous proof, as the proof is out of scope of the course.
   Hint: Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating f and g.

Solution:
Here, we first provide a "simplified" case, *i.e.*, we are constructing a dynamical system in which we pick equality for the inequality, that is:

$$\begin{cases} |f'(x)| = |g(x)|, \text{ and} \\ |g'(x)| = |f(x)|. \end{cases}$$

Without loss of generality, we may assume that f and g are non-negative before r, so the system becomes:

$$\begin{cases} f' = -g \\ g' = f \end{cases},$$

or equivalently, $\mathbf{y} = \begin{pmatrix} f \\ g \end{pmatrix}$ that $\mathbf{y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{y}$. Clearly, we observe the eigenvalues are $\pm\mathrm{i}$ as the polynomial is $\lambda^2 + 1 = 0$. Moreover, the eigenvectors for $\lambda_1 = \mathrm{i}$ is when $\begin{pmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$, in which we have $\boldsymbol{\xi} = y \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}$, and that solution is:

$$\mathbf{y} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} e^{\mathrm{i}x} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} (\cos x + \mathrm{i} \sin x) = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + \mathrm{i} \begin{pmatrix} \cos x \\ \sin x, \end{pmatrix}$$

and by conjugation, the solution should be:

$$\begin{pmatrix} f \\ g \end{pmatrix} = C_1 \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + C_2 \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}.$$

Note that with the given initial condition that $g(0) = 0$, this enforces $C_1 = 0$, thus $f(x) = C \cos x$ and $g(x) = C \sin x$, and we know that $f(r)$ is zero first at $r = \boxed{\pi/2}$.
The above version has some reasoning, but is not a rigorous proof at all, since this does not consider if r could be smaller than $\pi/2$. For students with interests, we provide the complete proof from the Putnam competition from Victor Lie, as follows.

Proof. Without loss of generality, we assume $f(x) > 0$ for all $x \in [0, r)$ as it is the first positive zero. By the fundamental theorem of calculus, we have:

$$|f'(x)| \le |g(x)| \le \left| \int_0^x g(s) ds \right| \le \int_0^x |g(s)| ds \le \int_0^t |f(s)| ds.$$

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Now, as we denote $F(x) = \int_0^x f(s)ds$, we have:

$$f'(x) + F(x) \geq 0 \text{ for } x \in [0, r].$$

For the sake of contradiction, we suppose $r < \pi/2$, then we have:

$$f'(x)\cos x + F(x)\cos x \geq 0 \text{ for } x \in [0, r].$$

Notice that the left hand side is the derivative of $f(x)\cos x + F(x)\sin x$, so an integration on $[y, r]$ gives:

$$F(r)\sin r \geq f(y)\cos y + F(y)\sin(y).$$

With some rearranging, we can have:

$$F(r)\sin r \sec^2 y \geq f(y)\sec y + F(y)\sin y \sec^2 y$$

Again, we integrate both sides with respect to y on $[0, r]$, which gives:

$$F(r)\sin^2 r \geq F(r),$$

and this is impossible, so we have a contradiction.
Hence we must have $r \geq \pi/2$, and since we have noted the solution $f(x) = C\cos x$ and $g(x) = C\sin x$, we have proven that $r = \pi/2$ is the smallest case. □

3. (Nilpotent Operator). Let M be a square matrix, M is defined to be *nilpotent* if:

$$M^k = 0 \text{ for some positive integer } k.$$

Similar to how we defined the exponential function analytically, the exponential function is also defined for matrices, let A be a square matrix, we define:

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i.$$

(a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

(b) Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
Hint: Use the differences of squares for matrices.

(c) If all the entries in N are rational, show that $\exp(N)$ has rational entries.

Solution:

(a) *proof of N is nilpotent.* Here, we want to do the matrix multiplication:

$$N^2 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N^3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we have shown that $N^3 = 0$, or the zero matrix, hence N is nilpotent. □

Then, we want to calculate the matrix exponential, that is:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}.$$

(b) *Proof.* Here, we recall the differences of squares still works when commutativity for multiplications fails, hence the we can still use it for matrix multiplication, namely, for all $m \in \mathbb{Z}^+$:

$$(\mathrm{Id}_n + N) \cdot (\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) = \mathrm{Id}_n - N^{2^{m+1}}$$

Since N is *nilpotent*, this implies that we have some k such that $N^\ell = 0$ for all $\ell \geq k$. Meanwhile, note that $2^\ell \geq \ell$ for all positive integer ℓ. (This can be proven by induction.) Therefore, we select $m + 1 \geq k$ so that $N^{2m+1} = 0$, and we have:

$$(\mathrm{Id}_n + N) \cdot \left[(\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) \right] = \mathrm{Id}_n,$$

thus $\mathrm{Id}_n + N$ is invertible. □

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(c) *Proof.* By the definition that N is nilpotent, we know that $N^m = 0$ for some finite positive integer m, hence, we can make the (countable) infinite sum into a finite sum:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \sum_{k=0}^{m} \frac{1}{k!} N^k,$$

thus all the entries are sum and non-zero divisions of rational numbers, while rational numbers are closed under addition and non-zero divisions, hence, all entries of $\exp(N)$ is rational. $\square$

Note that the elements of all n-by-n matrices can be considered as a *ring*, while *nilpotent* can be defined more generally for *rings*. We invite capable readers to investigate more properties of *nilpotent* elements of *rings* in the discipline of *Modern Algebra*.

4. (Convergence of Series.) As we dive into fundamentals of mathematics, it is inevitable to encounter *sequences* and their sums. Discuss about the following sequences if they converge or not. If they converge, find the explicit sum.

(a) $$\sum_{k=0}^{\infty} \frac{1}{k}.$$

(b) $$\sum_{k=0}^{\infty} \frac{1}{k!}.$$

(c) $$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!}.$$

Solution:

(a) Diligent readers should observe that $\sum_{k=0}^{\infty} 1/k$ is a harmonic series, hence it $\boxed{\text{diverges}}$. Otherwise, we can simply notice that:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{k} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \cdots \\
&\geq \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots \\
&= \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots\right) + \cdots \\
&= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots,
\end{aligned}$$

which diverges, hence our sequence $\sum_{k=0}^{\infty} 1/k$ must diverge.

(b) Here, we recall that the Taylor expansion of e^x at 0 is:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} e^0 (x-0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Evaluating the above equation at 1 gives that:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e^1 = \boxed{e},$$

in which the sequence $\boxed{\text{converges}}$.

(c) For this part, we want to note the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$ at 0 evaluated at $x = 1$ are, respectively:

$$\begin{array}{rclllllll}
e^1 &=& +\frac{1}{0!} & +\frac{1}{1!} & +\frac{1}{2!} & +\frac{1}{3!} & +\frac{1}{4!} & +\frac{1}{5!} & +\cdots \\
e^{-1} &=& +\frac{1}{0!} & -\frac{1}{1!} & +\frac{1}{2!} & -\frac{1}{3!} & +\frac{1}{4!} & -\frac{1}{5!} & +\cdots \\
\sin 1 &=& & +\frac{1}{1!} & & -\frac{1}{3!} & & +\frac{1}{5!} & -\cdots \\
\cos 1 &=& +\frac{1}{0!} & & -\frac{1}{2!} & & +\frac{1}{4!} & & -\cdots
\end{array}$$

Since the first series converges, we know that the later three series converges *absolutely*, so we are free to move around terms. Thus comparing vertically gives us that:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} = \boxed{\frac{e^1 - e^{-1}}{4} + \frac{\sin 1}{2}}.$$