PILOT

Problem Set 9

Differential Equations

Fall 2024

From this week, we are gradually transforming from linear system into non-linear system. As we generalize the cases of a linear system, always remind yourself with the notion of eigenspace and the geometries behind it. For non-linear system, you shall be thinking of how to remake into a linear system that we are familiar with.

1. (Repeated Eigenvalue). This problem investigates the case for repeated eigenvalues. First, we let the matrix $A \in \mathbb{R}^{2\times 2}$ be:

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here, we define the *algebraic multiplicity* of an eigenvalue as its multiplicity as a root to the characteristic polynomial, and the *geometric multiplicity* is the dimension of the eigenspace.

   (a) Find the eigenvalue and its corresponding eigenvector. State its algebraic and geometric multiplicity.

   (b) Find a the general solution to $\mathbf{x}' = A.\mathbf{x}$, where $\mathbf{x}' \in \mathbb{R}^2$.

Then, we consider the diagonal n-by-n matrices, that is matrices with entries only on the diagonal, which can be characterized as:

$$D = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

   (c) Show that the eigenvalues are exactly $a_1, \cdots, a_n$, and the algebraic multiplicity is exactly the same as geometric multiplicity for all eigenvalues.

   (d) Consider the linear system $\mathbf{x} = D.\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$, solve for the general solution for $\mathbf{x} = (x_1, \cdots, x_n)$. Explain why do not have to find the eigenvalues in this case.

2. (Complex Eigenvalue and Phase Portraits). Find the general solution and sketch a few phase portraits for:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.\mathbf{x}.$$

3. (Stability). Complete the following table for stability of dimension 2 linear systems.

Eigenvalues	**Type**	**Stability**
Eigenvalues are λ_1 and λ_2		
$0 < \lambda_1 < \lambda_2$		
$\lambda_1 < \lambda_2 < 0$		
$\lambda_1 < 0 < \lambda_2$		
$\lambda_1 = \lambda_2 > 0$		
$\lambda_1 = \lambda_2 < 0$		
Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$		
$\alpha > 0$		
$\alpha = 0$		
$\alpha < 0$		

4. (Fundamental Matrix). Let a system be defined as:

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}.$$

Find the general solution using a fundamental matrix.

Tip of the Week

Love thrifting, crafts, and good food? The Bmore Flea Market is open every Saturday until December from 10am-3pm at Broadway Market in Fells Point. Check out their Instagram @bmoreflea for more info!