Final Review Set: Solutions

Differential Equations

Fall 2024

1. Find the general solution for $y = y(t)$:
$$y' + 3y = t + e^{-2t},$$
then, describe the behavior of the solution as $t \to \infty$.

Solution:

Here, one could note that this differential equation is not separable but in the form of integrating factor problem, then we find the integrating factor as:
$$\mu(t) = \exp\left(\int_0^t 3ds\right) = \exp(3t).$$
By multiplying both sides with $\exp(3t)$, we obtain the equation:
$$y'e^{3t} + 3ye^{3t} = te^{3t} + e^{-2t}e^{3t}.$$
Clearly, we observe that the left hand side is the derivative after product rule for ye^{3t} and the right hand side can be simplified as:
$$\frac{d}{dt}[ye^{3t}] = te^{3t} + e^{t}.$$
Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:
$$\begin{aligned} ye^{3t} &= \int te^{3t}dt + \int e^{t}dt \\ &= \frac{te^{3t}}{3} - \int \frac{1}{3}e^{3t}dt + e^{t} + C \\ &= \frac{te^{3t}}{3} - \frac{e^{3t}}{9} + e^{t} + C. \end{aligned}$$
Eventually, we divide both sides by e^{3t} to obtain that:
$$y(t) = \boxed{\frac{t}{3} - \frac{1}{9} + e^{-2t} + Ce^{-3t}}.$$
Here, as $t \to \infty$, $y(t)$ diverges to $+\infty$ due to the term $t/3$.

2. Given an initial value problem:

$$\begin{cases} \dfrac{dy}{dt} - \dfrac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$

(a) Find the integrating factor $\mu(t)$.

(b) Solve for the particular solution for the initial value problem.

(c) Discuss the behavior of the solution as $t \to \infty$ for different cases of y_0.

Solution:

(a) As instructed, we look for the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t -\frac{3}{2}ds\right) = \boxed{\exp\left(-\frac{3}{2}t\right)}.$$

(b) With the integrating factor, we multiply both sides by $\mu(t)$ to obtain that:

$$y'e^{-3t/2} - \frac{3}{2}ye^{-3t/2} = 3te^{-3t/2} + 2e^te^{-3t/2}.$$

Clearly, we observe that the left hand side is the derivative after product rule for $ye^{-3t/2}$ and the right hand side can be simplified as:

$$\frac{d}{dt}\left[ye^{-3t/2}\right] = 3te^{-3t/2} + 2e^{-t/2}.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$\begin{aligned} ye^{-3t/2} &= \int 3te^{-3t/2}dt + \int 2e^{-t/2}dt \\ &= -2te^{-3t/2} + 2\int e^{-3t/2}dt - 4r^{-t/2} + C \\ &= -2te^{-3t/2} - \frac{4}{3}e^{-3t/2} - 4r^{-t/2} + C. \end{aligned}$$

Then, we divide both sides by $e^{-3t/2}$ to get the general solution:

$$y(t) = -2t - \frac{4}{3} - 4e^t + Ce^{3t/2}.$$

Given the initial condition, we have that:

$$y_0 = 0 - \frac{4}{3} - 4 + C,$$

which implies $C = 16/3 + y_0$, leading to the particular solution that:

$$y(t) = \boxed{-2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}}.$$

(c) We observe that:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[-2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}\right].$$

Note that the important terms are e^t and $e^{3t/2}$, we need to care the critical value $-16/3$:

- when $y_0 > -16/3$, $\boxed{y(t) \to \infty \text{ when } t \to \infty}$,
- when $y_0 \leq -16/3$, $\boxed{y(t) \to -\infty \text{ when } t \to \infty}$.

3. Suppose $f(x)$ is non-zero, let an initial value problem be:

$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

   (a) Show that the differential equation is **not** linear.

   For the next two questions, suppose $f(x) = \tan x$.

   (b) State, <u>without</u> justification, the open interval(s) in which $f(x)$ is continuous.

   (c)* Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

   Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

   (d)** Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

Solution:

(a) *Proof.* We can write the equation as:

$$F(x, y, y') := y' - \frac{xf(x)}{(y+1)(y-1)} = 0,$$

Note that:

$$F\big(x, (y+1), (y+1)'\big) = y' - \frac{xf(x)}{(y+2)y} \neq 1,$$

so the function is non-linear. □

(b) Here, we should consider that:

$$f(x) = \tan x = \frac{\sin x}{\cos x},$$

so the discontinuities are at when $\cos x = 0$, that is:

$$x \in \left\{ \frac{(2k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Hence, we have the intervals in which $f(x)$ being continuous as:

$$\boxed{\left\{ \left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2} \right) : k \in \mathbb{Z} \right\}}.$$

(c) *Proof.* Here, we want to write our equation in the standard form and obtain that:

$$y' := f(t, y) = \frac{x \tan x}{(y+1)(y-1)},$$

$$\frac{\partial f(t, y)}{\partial y} = -\frac{x \tan x \cdot 2y}{(y^2-1)^2}.$$

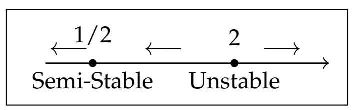
Clear, we note the discontinuities of y at $y = \pm 1$, and x demonstrated as above, thus we can form a rectangle $Q = (-\pi/2, \pi/2) \times (-1, 1)$ in which the initial condition $(0, 0) \in Q$ and $f(t, y)$ with $\partial_y f(t, y)$ are continuous on the interval. By the *existence and uniqueness theorem for non-linear case*, we know that there exists some δ such that there is a unique solution for $-\delta < x < \delta$. □

(d) If the condition in (c) does not hold, by contraposition, this implies that continuity must fail, *i.e.*, $\boxed{xf(x) \text{ must be discontinuous at } x = 0}$. Hence, some examples could be:

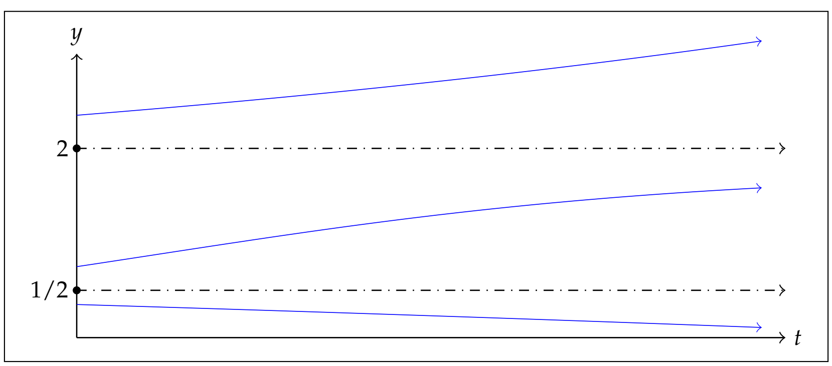
$$f(x) = \frac{1}{x^2}, \text{ or } \log x, \text{ or } \csc x, \text{ or } \chi_{\{0\}}(x) \text{ etc.}$$

4. An autonomous differential equation is given as follows:
$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \text{ where } t \geq 0 \text{ and } y \geq 0.$$
Draw a phase portrait and sketch a few solutions with different initial conditions.

Solution:
Recall from Pre-Calculus (or *Modern Algebra*) the following *Rational root test*:
Theorem: Rational Root Test. Let the polynomial:
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$
have integer coefficients $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$, then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p,q) = 1$ satisfies that $p | a_0$ and $q | a_n$. ⌟
From the theorem, we can note that if the equation has a rational root, it must be one of:
$$r = \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}.$$
By plugging in, one should notice that $y = 2$ is a root (one might also notice $1/2$ is a root as well, but we will get the step slowly), so we can apply the long division (dividing $y - 2$) to obtain that:
$$\frac{4y^3 - 12y^2 + 9y - 2}{y - 2} = 4y^2 - 4y + 1.$$
Clear, we can notice that the right hand side is a perfect square (else, you could use the quadratic formula) that:
$$4y^2 - 4y + 1 = (2y - 1)^2.$$
Thus, we now know that the roots are 2 and $1/2$ (multiplicity 2). Hence, the phase portrait is:



Correspondingly, we can sketch a few solutions (not necessarily in scale):



Note that the above **Theorem** can be generalized into the following manner (in ring theory):
Theorem: Rational Root Theorem. Let R be UFD, and polynomial:
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in R[x],$$
and let $r = p/q \in K(R)$ be a root of f with $p, q \in R$ and $\gcd(p,q) = 1$, then $p | a_0$ and $q | a_n$. ⌟
The proofs of the above **Theorems** are left as exercises to diligent readers. *Moreover, capable readers should attempt to prove that a polynomial of degree 3 with integer coefficients must have at least one rational root.*

5.* Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:

$$y'(x) = e^{2x} + y(x) - 1.$$

Solution:
First, we write the equation in the general form:

$$\frac{dy}{dx} + (1 - e^{2x} - y) = 0.$$

Now, we take the partial derivatives to obtain that:

$$\frac{\partial}{\partial y}[1 - e^{2x} - y] = -1,$$

$$\frac{\partial}{\partial x}[1] = 0.$$

Notice that the mixed partials are not the same, the equation is $\boxed{\text{not exact}}$.
Here, we choose the integrating factor as:

$$\mu(x) = \exp\left(\int_0^x \frac{\frac{\partial}{\partial y}[1 - e^{2s} - y] - \frac{\partial}{\partial s}[1]}{1} ds\right)$$
$$= \exp\left(\int_0^x -ds\right) = \exp(-x).$$

Therefore, our equation becomes:

$$(e^{-x})\frac{dy}{dx} + (e^{-x} - e^{x} - ye^{-x}) = 0.$$

After multiplying the integrating factor, it would be exact. *We leave the repetitive check as an exercise to the readers*.
Now, we can integrate to find the solution with a $h(y)$ as function:

$$\varphi(x, y) = \int (e^{-x} - e^{x} - ye^{-x})dx = -e^{-x} - e^{x} + ye^{-x} + h(y).$$

By taking the partial derivative with respect to y, we have:

$$\partial_y \varphi(x, y) = e^{-x} + h'(y),$$

which leads to the conclusion that $h'(y) = 0$ so $h(y) = C$.
Then, we can conclude that the solution is now:

$$\varphi(x, y) = -e^{-x} - e^{x} + ye^{-x} + C = 0,$$

which is equivalently:

$$y(x) = \boxed{\widetilde{C}e^{x} + 1 + e^{2x}}.$$

6. Solve the following second order differential equations for $y = y(x)$:

(a)
$$y'' + y' - 132y = 0.$$

(b)
$$y'' - 4y' = -4y.$$

(c)
$$y'' - 2y' + 3y = 0.$$

Solution:

(a) We find the characteristic polynomial as $r^2 + r - 132 = 0$, which can be trivially factorized into:
$$(r - 11)(r + 12) = 0,$$
so with roots $r_1 = 11$ and $r_2 = -12$, we have the general solution as:
$$y(x) = \boxed{C_1 e^{11x} + C_2 e^{-12x}}.$$

(b) We turn the equation to the standard form:
$$y'' - 4y' + 4 = 0.$$
We find the characteristic polynomial as $r^2 - 4r + 4 = 0$, which can be immediately factorized into:
$$(r - 2)^2 = 0,$$
so with roots $r_1 = r_2 = 2$ (repeated roots), we have the general solution as:
$$y(x) = \boxed{C_1 e^{2x} + C_2 x e^{2x}}.$$

(c) We find the characteristic polynomial as $r^2 - 2r + 3 = 0$, which the quadratic formula gives:
$$r = \frac{2 \pm \sqrt{2^2 - 4 \times 3}}{2} = 1 \pm \mathrm{i}\sqrt{2}$$
so with roots $r_1 = 1 + \mathrm{i}\sqrt{2}$ and $r_2 = 1 - \mathrm{i}\sqrt{2}$, we would have the solution:
$$y(x) = C_1 e^{(1+\mathrm{i}\sqrt{2})x} + C_2 e^{(1-\mathrm{i}\sqrt{2})x}.$$
To obtain real solution, we apply Euler's identity:
$$y_1(x) = e^x\big(\cos(\sqrt{2}x) - \mathrm{i}\sin(\sqrt{2}x)\big) \text{ and } y_2(x) = e^x\big(\cos(-\sqrt{2}x) - \mathrm{i}\sin(-\sqrt{2}x)\big).$$
By the *principle of superposition*, we can linearly combine the solutions to be different solutions, so we have:
$$\widetilde{y_1}(x) = \frac{1}{2}(y_1 + y_2) = e^x \cos(\sqrt{2}x),$$
$$\widetilde{y_2}(x) = \frac{1}{2}(y_2 - y_1) = e^x \sin(\sqrt{2}x).$$
One can verify that $\widetilde{y_1}$ and $\widetilde{y_2}$ are linearly independent by taking Wronskian,*i.e.*:
$$\begin{aligned} \mathrm{W}[\widetilde{y_1}, \widetilde{y_2}] &= \det\begin{pmatrix} e^x \cos(\sqrt{2}x) & e^x \sin(\sqrt{2}x) \\ e^x \cos(\sqrt{2}x) - \sqrt{2}e^x \sin(\sqrt{2}x) & e^x \sin(\sqrt{2}x) + \sqrt{2}e^x \cos(\sqrt{2}x) \end{pmatrix} \\ &= \sqrt{2}e^{2x}\cos^2(\sqrt{2}x) + \sqrt{2}e^{2x}\sin^2(\sqrt{2}x) = \sqrt{2}e^{2x} \neq 0. \end{aligned}$$
Now, they are linearly independent, so we have the general solution as:
$$y(x) = \boxed{C_1 e^x \cos(\sqrt{2}x) + C_2 e^x \sin(\sqrt{2}x)}.$$

7. Given a differential equation for $y = y(t)$ being:

$$t^3 y'' + ty' - y = 0.$$

(a) Verify that $y_1(t) = t$ is a solution to the differential equation.

(b)* Find the full set of solutions using reduction of order.

(c) Show that the set of solutions from part (b) is linearly independent.

Solution:

(a) *Proof.* We note that the left hand side is:

$$t^3 y_1'' + ty_1' - y_1 = t^3 \cdot 0 + t \cdot 1 - t = t - t = 0.$$

Hence $y_1(t) = t$ is a solution to the differential equation. □

(b) By reduction of order, we assume that the second solution is $y_2(t) = tu(t)$, then we plug $y_2(t)$ into the equation to get:

$$2t^3 u'(t) + t^4 u''(t) + tu(t) + t^2 u'(t) = t^4 u''(t) + (2t^3 + t^2)u'(t) = 0.$$

Here, we let $\omega(t) = u'(t)$ to get a first order differential equation:

$$t^2 \omega'(t) = (-2t - 1)\omega(t).$$

Clearly, this is separable, and we get that:

$$\frac{\omega'(t)}{\omega(t)} = -\frac{2t+1}{t^2} = -\frac{2}{t} - \frac{1}{t^2},$$

which by integration, we have obtained that:

$$\log\left(\omega(t)\right) = -2\log t + \frac{1}{t} + C.$$

By taking exponentials on both sides, we have:

$$\omega(t) = \exp\left(-2\log t + \frac{1}{t} + C\right) = \widetilde{C} e^{1/t} \cdot \frac{1}{t^2}.$$

Recall that we want $u(t)$ instead of $\omega(t)$, so we have:

$$u(t) = \int \omega(t)dt = \widetilde{C} \int e^{1/t} \cdot \frac{1}{t^2} dt = -\widetilde{C} e^{1/t} + D.$$

By multiplying t, we obtain that:

$$y_2 = -\widetilde{C} t e^{1/t} + Dt,$$

where Dt is repetitive in y_1, so we get:

$$y(t) = \boxed{C_1 t + C_2 t e^{1/t}}.$$

(c) *Proof.* We calculate Wronskian as:

$$W[t, te^{1/t}] = \det \begin{pmatrix} t & te^{1/t} \\ 1 & e^{1/t} - \frac{e^{1/t}}{t} \end{pmatrix} = -e^{1/t} \neq 0,$$

hence the set of solutions is linearly independent. □

8.** Given the following second order initial value problem:

$$\begin{cases} \frac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \\ \frac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
Hint: Consider the interval in which the solution is unique.

Solution:
Note that I deliberately messed up with all the messy functions. Not only haven't I found a solution to the system, Wolfram cannot have an elementary solution as well. Hence, we need to think, alternatively, on some theorems.

Proof. Here, we suppose that $y(x)$ is a solution, and we want to show that $y(2-x)$ is also a solution. First we note that we can think of taking the derivatives of $y(2-x)$, by the chain rule:

$$\frac{d}{dx}[y(2-x)] = -y'(2-x),$$
$$\frac{d^2}{dx^2}[y(2-x)] = y''(2-x).$$

Now, if we plug in $y(2-x)$ into the system of equations, we have:

- First, for the differential equation, we have:

$$\begin{aligned} \frac{d^2}{dx^2}[y(x-2)] + \cos(1-x)y(x-2) &= y''(2-x) + \cos(x-1)y(2-x) \\ &= y''(2-x) + \cos\left(1-(2-x)\right)y(2-x) \\ &= y''(x) + \cos(1-x)y(x) \\ &= x^2 - 2x + 1 = (x-1)^2 = (1-x)^2 \\ &= \left((2-x)-1\right)^2 = (2-x)^2 - 2(2-x) + 1. \end{aligned}$$

- For the initial conditions, we trivially have that:

$$y(1) = y(2-1) \text{ and } y'(1) = y'(2-1).$$

Hence, we have shown that $y(2-x)$ is a solution if $y(x)$ is a solution.
Again, we observe the original initial value problem that:

$$\cos(1-x) \text{ and } x^2 - 2x + 1 \text{ are continuous on } \mathbb{R}.$$

Therefore, by the *existence and uniqueness theorem for second order linear case*, there could be only one solution, which forces that:

$$y(x) = y(2-x),$$

so the solution is symmetric about $x = 1$, as desired. □

9. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.
   (a) $$y'' + 2y' + y = e^{-t}.$$
   (b) $$y'' + y = \tan t.$$

Solution:

(a) First, we look for homogeneous solution, *i.e.*, $y'' + 2y' + y = 0$, whose characteristic equation is:
$$r^2 + 2r + 1 = (r+1)^2 = 0,$$
with root(s) being -1 with multiplicity of 2, so the general solution to homogeneous case is:
$$y_g(t) = C_1 e^{-t} + C_2 t e^{-t}.$$
Notice that the non-homogeneous part is e^{-t}, but we have e^{-t} and te^{-t} as general solutions already, so we have our guess of particular solution as:
$$y_p(t) = At^2 e^{-t}.$$
By taking the derivatives, we have:
$$y_p'(t) = A(2te^{-t} - t^2 e^{-t}) \qquad \text{and} \qquad y_p''(t) = A(2e^{-t} - 4te^{-t} + t^2 e^t).$$
We simply plug in the particular solution, so we have:
$$\begin{aligned} A(2e^{-t} - 4te^{-t} + t^2 e^t) + 2A(2te^{-t} - t^2 e^{-t}) + At^2 e^{-t} &= e^{-t} \\ 2Ae^{-t} &= e^{-t} \\ A &= \frac{1}{2}. \end{aligned}$$
Hence, our solution to the non-homogeneous case is:
$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}}.$$

(b) Here, we still look for homogeneous solutions, *i.e.*, $y'' + y = 0$, whose characteristic equation is:
$$r^2 + 1 = 0,$$
with roots $\pm \mathrm{i}$. Since we are dealing with real valued functions, we have the general solution as:
$$y_g = C_1 \sin t + C_2 \cos t.$$
Note that $\tan t$ does not work with undetermined coefficients, we must use the variation of parameters, the Wronskian of our solution is:
$$\mathrm{W}[\sin t, \cos t] = \det \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} = -\sin^2 t - \cos^2 t = -1.$$

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Now, we may use the formula, namely getting the particular solution as:

$$\begin{aligned}
y_p &= \sin t \int \frac{-\cos t \cdot \tan t}{-1} dt + \cos t \int \frac{\sin t \cdot \tan t}{-1} dt \\
&= \sin t \int \sin t dt - \cos t \int \frac{\sin^2 t}{\cos t} dt \\
&= \sin t(-\cos t + C) - \cos t \int \frac{1 - \cos^2 t}{\cos t} dt \\
&= -\sin t \cos t + \cancel{C \sin t} - \cos t \left(\int \sec t dt - \int \cos t dt \right) \\
&= -\sin t \cos t - \cos t \left(\log |\sec t + \tan t| - \sin t + C \right) \\
&= -\sin t \cos t + \sin t \cos t - \cancel{C \cos t} - \cos t \log |\sec t + \tan t| \\
&= -\cos t \log |\sec t + \tan t|.
\end{aligned}$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 \sin t + C_2 \cos t - \cos t \log |\sec t + \tan t|}.$$

10. Solve for the general solution to the following higher order ODE.

(a)
$$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

(b)**
$$\frac{d^4y}{dx^4} + y = 0.$$

Hint: Consider the 8-th root of unity, *i.e.*, ζ_8, and verify which roots satisfies the polynomial.

Solution:

(a) Note that we obtain the characteristic equation as:
$$4r^4 - 24r^3 + 45r^2 - 29r + 6 = 0.$$
To obtain our roots, we use the **Rational Root Theorem**, so if the characteristic equation has any rational root, it must have been one (or more) of the following:
$$\pm 1, \pm 2, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}.$$
From plugging in the values, we notice that 2 and 3 are roots of the characteristic equation, by division, we have:
$$\frac{4r^4 - 24r^3 + 45r^2 - 29r + 6}{(r-2)(r-3)} = 4r^2 - 4r + 1 = (2r-1)^2.$$
Now, we know that the roots are 2, 3, and $1/2$ with multiplicity 2, thus the solution to the differential equation is:
$$y(x) = \boxed{C_1e^{2x} + C_2e^{3x} + C_3e^{x/2} + C_4xe^{x/2}}.$$
Again, we invite readers to verify the ***Rational Root Theorem*** *(c.f. Problem 4).*

(b) For this general solution, we trivially obtain that the characteristic polynomial is:
$$r^4 + 1 = 0.$$
Recall that the root of unity address for the case when $r^n = 1$, so we consider the 8th root of unity, in which $(\zeta_8)^8 = 1$. Now, recall **Euler's Identity** and **deMoivre's formula**, we note that only the odd powers of the 8th root of unity satisfies that $r^4 = -1$, namely, are:
$$\zeta_8 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2},$$
$$\zeta_8^3 = \cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2},$$
$$\zeta_8^5 = \cos\left(\frac{5\pi}{4}\right) + \mathrm{i}\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2},$$
$$\zeta_8^7 = \cos\left(\frac{7\pi}{4}\right) + \mathrm{i}\sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}.$$
Also, we note that ζ_8 and ζ_8^7 are complex conjugates, whereas ζ_8^3 and ζ_8^5 are complex conjugates, so we can linearly combine them to obtain the set of linearly independent solutions, *i.e.*:
$$y(x) = \boxed{\begin{aligned} & e^{(\sqrt{2}/2)x}\left[C_1\cos\left(\frac{\sqrt{2}}{2}x\right) + C_2\sin\left(\frac{\sqrt{2}}{2}x\right)\right] \\ + \; & e^{-(\sqrt{2}/2)x}\left[C_3\cos\left(\frac{\sqrt{2}}{2}x\right) + C_4\sin\left(\frac{\sqrt{2}}{2}x\right)\right]\end{aligned}}.$$

11. Let a third order differential equation be as follows:

$$\ell\big[y(t)\big] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$

Let $\ell\big[y(t)\big] = 0$ be trivial initially.

(a) Find the set of all linearly independent solutions.

Then, assume that $\ell\big[y(t)\big]$ is non-trivial.

(b) Find the particular solution to $\ell\big[y(t)\big] = \sin t$.

(c) Find the particular solution to $\ell\big[y(t)\big] = e^{-t}$.

(d)* Suppose that $\ell\big[y_1(t)\big] = f(t)$ and $\ell\big[y_2(t)\big] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell\big[y_3(t)\big] = f(t) + g(t)$.

Solution:

(a) Note that the characteristic polynomial can be factorized as perfect cubes:

$$r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0,$$

its roots are $r = -1$ with multiplicity 3, so the general solution is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}}.$$

Here, the readers are invited to check, by ***Wronskian****, that set of solutions are linearly independent.*

(b) First, we want to make our guess of particular solution as:

$$y_p(t) = A\sin t + B\cos t,$$

and by taking the derivatives, we have:

$$y_p'(t) = A\cos t - B\sin t, \qquad y_p''(t) = -A\sin t - B\cos t, \text{ and } \qquad y_p'''(t) = -A\cos t + B\sin t.$$

Then, we want to plug in the results into the equation, so:

$$\begin{aligned}\ell\big[y_p(t)\big] &= (-A\cos t + B\sin t) + 3(-A\sin t - B\cos t) + 3(A\cos t - B\sin t) + A\sin t + B\cos t\\ &= (B - 3A - 3B + A)\sin t + (-A - 3B + 3A + B)\cos t\\ &= (-2A - 2B)\sin t + (2A - 2B)\cos t.\end{aligned}$$

Therefore, we can obtain the system that:

$$\begin{cases} -2A - 2B = 1,\\ 2A - 2B = 0,\end{cases}$$

which reduces to $A = -1/4$ and $B = -1/4$, so the solution is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t} - \frac{1}{4}\sin t - \frac{1}{4}\cos t}.$$

(c) Here, note that e^{-t}, te^{-t}, and t^2e^{-t} are the solutions to homogeneous case, our guess, then, is:

$$y_p(t) = At^3e^{-t},$$

and by taking the derivatives, we have:

$$y_p'(t) = 3At^2e^{-t} - At^3e^{-t}, \qquad y_p''(t) = 6Ate^{-t} - 6At^2e^{-t} + At^3e^{-t}, \qquad \text{and}$$
$$y_p'''(t) = 6Ae^{-t} - 18Ate^{-t} + 9At^2e^{-t} - At^3e^{-t}.$$

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When we plug the derivatives back to the solutions, we note that:

$$\begin{aligned}\ell[y_p(t)] &= (6Ae^{-t} - 18Ate^{-t} + 9At^2e^{-t} - At^3e^{-t}) \\ &\quad + 3(6Ate^{-t} - 6At^2e^{-t} + At^3e^{-t}) + 3(3At^2e^{-t} - At^3e^{-t}) + (At^3e^{-t}) \\ &= 6Ae^{-t},\end{aligned}$$

which reduces to $A = 1/6$, so the solution is:

$$y(t) = \boxed{C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} + \frac{1}{6}t^3e^{-t}}.$$

(d) *Proof.* Here, one should note that the derivative operator is linear, so we have that:

$$\begin{aligned}\ell[y_1(t) + y_2(t)] &= \frac{d^3}{dt^3}[y_1(t) + y_2(t)] + 3\frac{d^2}{dt^2}[y_1(t) + y_2(t)] + 3\frac{d}{dt}[y_1(t) + y_2(t)] + [y_1(t) + y_2(t)] \\ &= y_1'''(t) + 3y_1''(t) + 3y_1'(t) + y_1(t) + y_2'''(t) + 3y_2''(t) + 3y_2'(t) + y_2(t) \\ &= f(t) + g(t),\end{aligned}$$

as desired. □

12. Let a system of differential equations be defined as follows, find the general solutions to the equation:

(a) $$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^2,$$

(b)* $$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3.$$

Solution:

(a) The question should be trivial, we first find the eigenvalues for the equation, *i.e.*:

$$\det \begin{pmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = 0,$$

which is $(3-\lambda)(2-\lambda) = 0$, that is $\lambda_1 = 3$ and $\lambda_2 = 2$. Then, we look for the eigenvectors.

- For $\lambda_1 = 3$, we have $\begin{pmatrix} 3-3 & 0 \\ 0 & 2-3 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $\boldsymbol{\xi}_1 = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- For $\lambda_2 = 2$, we have $\begin{pmatrix} 3-2 & 0 \\ 0 & 2-2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$, which is $\boldsymbol{\xi}_2 = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

(b) Again, we first find the eigenvalues of the equation, *i.e.*:

$$\det \begin{pmatrix} 1-\lambda & 0 & 4 \\ 1 & 1-\lambda & 3 \\ 0 & 4 & 1-\lambda \end{pmatrix} = 0,$$

which is $(1-\lambda)^3 + 16 - 12(1-\lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = -(\lambda+1)^2(\lambda-5) = 0$. Hence, the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Now, we look for eigenvectors.

- For $\lambda_1 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $x \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$.
- For $\lambda_2 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} 4x \\ x+1 \\ -2x-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
- For $\lambda_3 = 5$, we have $\begin{pmatrix} -4 & 0 & 4 \\ 1 & -4 & 3 \\ 0 & 4 & -4 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$, which is $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + C_2 \left(te^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) + C_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}.$$

13. Solve the following initial value problem for the system of equations:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution:

Here, we first find the eigenvalues for the matrix, that is:

$$\det \begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} = 0.$$

Therefore, the polynomial is $(1-\lambda)(-7-\lambda)+16 = (\lambda+3)^2 = 0$, hence the eigenvalues is $\lambda_1 = \lambda_2 = -3$. Then, we look for the eigenvectors.

- For $\lambda_1 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $\boldsymbol{\xi}_1 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} x \\ x - 1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$.

Hence, the general solution is:

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} \right).$$

By the initial condition, we have $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, so:

$$\mathbf{x}(0) = \begin{pmatrix} C_1 + 0 \\ C_1 - C_2/4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore, $C_1 = 3$ and $C_2 = 4$, so the particular solution is:

$$\mathbf{x}(t) = \boxed{\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} te^{-3t}}.$$

14. Let a system of differential equations of $x_i(t)$ be as follows:

$$\begin{cases} x_1' = 3x_1 + 2x_2, & x_1(1) = 0, \\ x_2' = x_1 + 4x_2, & x_2(1) = 2. \end{cases}$$

(a) Solve for the solution to the initial value problem.

(b) Identify and describe the stability at equilibrium(s).

Solution:

(a) Here, we denote $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^\intercal$, so our system becomes:

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Here, the eigenvalues are solutions to:

$$\det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} = 0,$$

which simplifies to $\lambda^2 - 7\lambda + 10 = 0$, and further gives $\lambda_1 = 2$, $\lambda_2 = 5$. Then, we look for eigenvectors of the matrix:

- For $\lambda_1 = 2$, we have $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which gives that $\boldsymbol{\xi}_1 = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = 5$, we have $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$, which gives that $\boldsymbol{\xi}_2 = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Now, the general solution must be:

$$\mathbf{x} = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t},$$

and by plugging in the initial condition, we have:

$$\begin{cases} -2C_1 e^2 + C_2 e^5 = 0, \\ C_1 e^2 + C_2 e^5 = 2. \end{cases}$$

In which the solution is $C_1 = \frac{2}{3e^2}$ and $C_2 = \frac{4}{3e^5}$, so the solution is:

$$\boxed{\begin{cases} x_1 = -\frac{4}{3} e^{2t-2} + \frac{4}{3} e^{5t-5}, \\ x_2 = \frac{2}{3} e^{2t-2} + \frac{4}{3} e^{5t-5}. \end{cases}}$$

(b) Now, we consider the equilibrium at $\mathbf{x} = \begin{pmatrix} 0 & 0 \end{pmatrix}^\intercal$, in which we note that both eigenvalues are positive, meaning that this is an $\boxed{\text{unstable node}}$.

15. Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle θ (counter-clockwise), then:

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a)* Show that $M^\intercal = M^{-1}$.

(b)** Let $\theta = 2\pi/k$ be fixed, where k is an integer. Find the least positive integer n such that $M^n = \mathrm{Id}_2$. Here, n is called the *order* of M.

Hint: Consider the rotational matrix geometrically, rather than arithmetically.

Solution:

(a) *Proof.* Here, we recall the method of inverting a matrix:

$$M^{-1} = \frac{1}{\det M}\begin{pmatrix} \cos\theta & -(-\sin\theta) \\ -\sin\theta & \cos\theta \end{pmatrix} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = M^\intercal.$$

□

(b) Look, we want to analyze this geometrically, if $\theta = 2\pi/k$, then that implies that M is a counter-clockwise rotation of $2\pi/k$, and since a full revolution is 2π, this implies a rotation of k times will make restore to the original vector, *i.e.*, $M^k = \mathrm{Id}_2$. Moreover, for any positive integer less than k, we cannot rotate back to 2π, which implies that the order of M is $\boxed{k}$.

16. Let a non-linear system be:

$$\frac{dx}{dt} = x - y^2 \text{ and } \frac{dy}{dt} = x + x^2 - 2y.$$

Verify that $(0,0)$ is a critical point and classify its type and stability.

Solution:

proof that $(0,0)$ is critical point. The verification of $(0,0)$ being a critical point is trivial. We check that dx/dt and dy/dt evaluated at $(0,0)$ are:

$$\left.\frac{dx}{dt}\right|_{(0,0)} = 0 \text{ and } \left.\frac{dy}{dt}\right|_{(0,0)} = 0,$$

and hence $(0,0)$ is a critical point. □

In particular, denoting $\mathbf{x} = (x, y)$, we verify the linear approximation as:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x},$$

and we note that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, and by:

$$\lambda_2 < 0 < \lambda_1,$$

we know that we have a $\boxed{\text{unstable saddle point}}$ at $(0,0)$.

17. Let a system of non-linear differential equations be defined as follows:

$$\begin{cases} x' = 2x + 3y^2, \\ y' = x + 4y^2. \end{cases}$$

Find all equilibrium(s) and classify their stability locally.

Solution:

Here, we note that the equilibrium(s) is achieved if and only if $x' = y' = 0$, that is:

$$\begin{cases} 2x + 3y^2 = 0, \\ x + 4y^2 = 0. \end{cases}$$

In particular, we consider $z = y^2$, so we have a system of linear equations, that is:

$$\begin{cases} 2x + 3z = 0, \\ x + 4z = 0. \end{cases}$$

Meanwhile, the above system simplifies to $x = y = 0$, hence the only equilibrium is at $(x, y) = (0, 0)$. Then, we consider the system locally, denoting $\mathbf{x} = (x, y)$, that is:

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x},$$

where the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 0$. Note that one eigenvalue is zero and the other is positive, then the critical point is $\boxed{\text{unstable}}$.

18. Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:

$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$

Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.

(a) Find the set of all equilibrium(s) for $\mathbf{x}$.

(b) Find the set in which the equilibrium(s) is locally linear.

Now, $F : \mathbb{R}^2 \to \mathbb{R}$ is not necessarily well-behaved.

(c)** Construct a function F such that $\mathbf{x}$ has a equilibrium that is <u>not</u> locally linear.
Hint: Consider the condition in which a non-linear system is locally linear.

Solution:

(a) Here, we note that the equilibrium is when $F(\mathbf{x}) = 0$, *i.e.*, $\sin x_1 + \csc(3x_2) = 0$. Here, we note that the image of $\sin x_1$ is $[-1, 1]$ and the image of $\sec(3x_2)$ is $(-\infty, -1] \sqcup [1, \infty)$, this implies that $\sin x_1 + \sec(3x_2)$ is zero only if $\sin x_1 = \pm 1$ and $\sec(3x_2) = \mp 1$, correspondingly.
First, we consider the set in which x_1 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Correspondingly, we consider the set in which x_2 is -1, that is:

$$\left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Then, we consider the set in which x_1 is -1, that is:

$$\left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Likewise, we consider the set in which x_2 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Therefore, set theoretically, we have the set of all equilibriums as:

$$\boxed{\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}}.$$

(b) Note that $\sin x_1$ is (twice) differentiable over the entire domain $\mathbb{R}$ and $\csc(3x_2)$ is (twice) differentiable on all neighborhoods when $\csc(3x_2)$ is ∓ 1, hence the partial derivatives of $F(\mathbf{x})$ with respect to x_1 or x_2 are (twice) differentiable on the neighborhood on all equilibriums, hence the set in which the equilibrium(s) is locally linearly is the same from part (a), namely:

$$\boxed{\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}}.$$

(c) Clearly, we must enforce that $F(\mathbf{x})$ is $\boxed{\text{not twice differentiable}}$ with some partial derivatives near the equilibrium point(s). One trivial example could be using the absolute value, such as $F(\mathbf{x}) = |x_1| + |x_2|$, where $(0, 0)$ is a equilibrium but it is <u>not differentiable</u>.
For capable readers, we invite them to look for more functions, such as the Weierstrass Function, a continuous function that is *nowhere* differentiable:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cos(3^k x).$$

19. Let a system of (x, y) be functions of variable t, and they have the following relationship:

$$x' = (1+x)\sin y \text{ and } y' = 1 - x - \cos y.$$

(a) Identify the corresponding linear system.

(b) Evaluate the stability for the equilibrium at $(0,0)$ by showing it is locally linear.

Solution:

(a) Here, since we can write:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x)\sin y \\ 1-\cos y \end{pmatrix},$$

this implies that the linear system is:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}.$$

(b) $(0,0)$ *is locally linear.* We find the Jacobian Matrix, that is:

$$\mathbf{J} = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix}.$$

As we evaluate $\mathbf{J}$ at $(0,0)$ and take its determinant, we have:

$$\det\left(\mathbf{J}\big|_{(0,0)}\right) = \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 \neq 0.$$

Hence, the $(0,0)$ is locally linear. □

Note that we have found the linear system in part (a), whose eigenvalues are $\lambda_1 = \lambda_2 = 0$. Since $x' = 0$, it indicates that x is a constant, whereas for $y' = -x$ indicates that it will be a $\boxed{\text{unstable almost everywhere}}$ for all neighborhoods of $(0,0)$.

In particular, readers could illustrate the "slope field" for the linear system in (a), and they should notice that except for $x = 0$ being entirely stable, all other trajectory would shift vertically at a constant rate. However, the line $x = 0$ will always be insignificant enough (having *Lebesgue measure* 0), hence we claim that it is unstable almost everywhere. For interested readers, please explore *Lebesgue measure* as a way to determine how large a subset is in Euclidean space.

20.** Let a locally linearly system be defined as:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x} + \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function. Find the necessary condition(s) in which the equilibrium(s) have a stable *center* in linear system. Then, state the stability and type (if possible).
Hint: Consider the solution for the linear case or matrix exponential.

Solution:
Without loss of generality, we assume that the system of $\mathbf{x}$ has equilibrium(s), else the statement is vacuously true. Now, we start to evaluate the additional conditions with such assumption:

(i) Note that the system needs to be locally linearly, *i.e.*, we must have $\boxed{\mathbf{f}(\mathbf{x}) \text{ being twice differentiable with respect to partial derivatives}}$.

(ii) Moreover, we need to worry about the linear system to have a *stable center*, that is:

$$\mathbf{x}' = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x}.$$

Note that the eigenvalues would be the solutions to $(\lambda - r)^2 + \mu^2 = 0$, that is $r = \lambda \pm i\mu$, which is a pair of complex conjugates. Here, in to be stable, we want $\lambda \leq 0$, and for center, this forces $\boxed{\lambda = 0}$.

Note that even the linear system is a stable center, the stability of the non-linear system is $\boxed{\text{indeterminate}}$, and the type is $\boxed{\text{center or spiral point}}$.

21. Given the a system of differential equations as follows:

$$\begin{cases} x' = x - y - x(x^2 + y^2), \\ y' = x + y - y(x^2 + y^2). \end{cases}$$

Find the limit cycle of the system, classify the critical points, and sketch a phase portrait of the system.

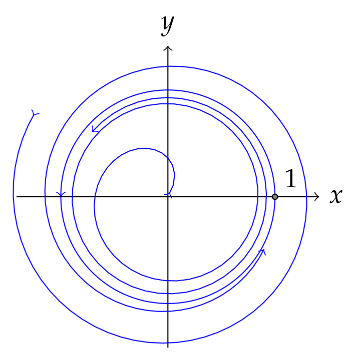
Solution:
For this problem, we recall the formula converting between polar coordinates and Cartesian coordinates:

$$\begin{cases} x = r\cos\theta, & y = r\sin\theta, \\ rr' = xx' + yy', & r^2\theta' = xy' - yx'. \end{cases}$$

Now, we are able to convert the system as:

$$\begin{aligned} rr' &= x\big(x - y - x(x^2+y^2)\big) + y\big(x + y - y^2(x^2+y^2)\big) \\ &= x^2 - xy - x^2(x^2+y^2) + xy + y^2 - y^2(x^2+y^2) \\ &= x^2 + y^2 - (x^2+y^2)(x^2+y^2) = r^2 - r^4. \\ r' &= r - r^3 = r(1-r^2) = r(1+r)(1-r). \\ r^2\theta' &= x\big(x + y - y(x^2+y^2)\big) - y\big(x - y - x(x^2+y^2)\big) \\ &= x^2 + xy - xy(x^2+y^2) - xy + y^2 + xy(x^2+y^2) \\ &= x^2 + y^2 = r^2. \\ \theta' &= 1. \end{aligned}$$

Therefore, the system is having limit cycle at $\boxed{r = 0 \text{ and } r = 1}$. Since $r' > 0$ for $r \in (0,1)$ and $r' < 0$ for $r \in (1,\infty)$, thus the limit cycle $r = 0$ is unstable and the limit cycle $r = 1$ is stable. The phase portrait can be illustrated as follows:



22. Consider the following series. Identify if such series converges. Compute the limit if the series converges.

(a)
$$\sum_{k=0}^{\infty} \frac{n!}{2^n}.$$

(b)*
$$\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!}.$$

(c)
$$\sum_{k=0}^{\infty} \frac{x^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!}.$$

Solution:

(a) Here, we do the ratio test:
$$\lim_{n\to\infty} \frac{(n+1)!/2^{n+1}}{n!/2^n} = \lim_{n\to\infty} \frac{n+1}{2} = \infty.$$
Hence, the series diverges.

As a side note, if you have seen some algorithms in computer science, you might have seen that:
$$\mathcal{O}(2^n) \subset \mathcal{O}(n!).$$
which is the asymptotic behavior of complexity.

(b) For the question, we expand all the terms of the power series for e^x, e^{-x}, $\sin x$, and $\cos x$ out (since they converge absolutely), explicitly as:
$$\begin{array}{lcccccccc}
e^x & \sim & +\frac{x^0}{0!} & +\frac{x^1}{1!} & +\frac{x^2}{2!} & +\frac{x^3}{3!} & +\frac{x^4}{4!} & +\frac{x^5}{5!} & +\cdots \\
e^{-x} & \sim & +\frac{x^0}{0!} & -\frac{x^1}{1!} & +\frac{x^2}{2!} & -\frac{x^3}{3!} & +\frac{x^4}{4!} & -\frac{x^5}{5!} & +\cdots \\
\sin x & \sim & & +\frac{x^1}{1!} & & -\frac{x^3}{3!} & & +\frac{x^5}{5!} & -\cdots \\
\cos x & \sim & +\frac{x^0}{0!} & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & -\cdots
\end{array}$$
By some arithmetics, one should notice that:
$$\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} = \boxed{\frac{e^x - e^{-x}}{4} + \frac{\sin x}{2}}.$$
Hence, the power series converges.

(c) For this sequence, we note that:
$$\sum_{k=0}^{\infty} \frac{x^{4k}}{(4k)!} + \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \boxed{\cos x}.$$
Thus the power series converges.

23. Use the *series expansions* to find the solutions to the following differential equation:

$$y'' + 3y' = 0.$$

Solution:

Here, we note that we have constant coefficients, so they are automatically analytic. Now, we take $x_0 = 0$, and assume that our solution is in the form that:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Now, by the assumption that the series converges absolute, we take differentiate the terms twice, which gives that:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

and:

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

With the derivative, we plug it back into the differential equations, that is:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 3 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 0.$$

By the term-wise addition, we have:

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + 3(n+1) a_{n+1} \right] x^n = 0.$$

Given that the sequence is equivalently zero, then we have the relation as:

$$(n+2)(n+1) a_{n+2} + 3(n+1) a_{n+1} = 0,$$

which is equivalently:

$$a_{n+2} = -\frac{3(n+1) a_{n+1}}{(n+2)(n+1)} = -\frac{3 a_{n+1}}{n+2}.$$

So we can simplify the recurrence relationship as:

$$\boxed{a_{n+1} = -\frac{3a_n}{n+1}} \text{ for } n \geq 1.$$

Now, since this differential equation has order 2, we let the first two coefficients fixed, that is a_0 and a_1, then we can form the rest of the coefficients as:

$$a_2 = -\frac{3a_1}{2}, \qquad a_3 = -\frac{3a_2}{3} = \frac{3^2 a_1}{3!}, \qquad a_4 = -\frac{3a_3}{4} = -\frac{3^3 a_1}{4!}, \qquad \cdots.$$

Thus, the general form is:

$$a_n = (-1)^{n-1} \frac{3^{n-1} a_1}{n!} \text{ for } n \geq 1.$$

Thus, the solution for this problem is:

$$y(x) = a_0 + a_1 \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n!} x^n = \widetilde{a}_0 + 1 + a_1 \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n!} x^n.$$

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Recall that the power series of e^x is:

$$e^x \sim \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Thus, we have:

$$e^{-3x} \sim \sum_{n=0}^{\infty} \frac{1}{n!} (-3x)^n = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n.$$

Now, we can also switch to $\widetilde{a_1}$ as $a_1 = -3\widetilde{a_1}$, so we have:

$$y(x) = \widetilde{a_0} + \widetilde{a_1} \sum_{n=1}^{\infty} \frac{(-3)^n}{n!} x^n = \boxed{\widetilde{a_0} + \widetilde{a_1} e^{-3x}}.$$

24. Use the *Euler's equation* to find the solution to the following differential equations:

(a) $$x^2y'' + 5xy' + 4y = 0.$$

(b) $$5x^2y'' + 3xy' + 7y = 0.$$

Solution:

(a) Here, our characteristic equation is:

$$0 = r(r-1) + 5r + 4 = r^2 + 4r + 4 = (r+2)^2,$$

whose repeated root is -2, so the solution is:

$$y(x) = \boxed{c_1|x|^{-2} + c_2 \log|x| \cdot |x|^{-2}}.$$

(b) Here, we can write the equations as:

$$x^2y'' + \frac{3}{5}xy' + \frac{7}{5}y = 0.$$

Thus, our characteristic equation is:

$$0 = r(r-1) + \frac{3}{5}r + \frac{7}{5} = r^2 - \frac{2}{5}r + \frac{7}{5}.$$

Now, we have the roots as:

$$r = \frac{\frac{2}{5} \pm \sqrt{\frac{4}{25} - \frac{28}{5}}}{2} = \frac{1}{5} \pm \sqrt{\frac{1}{25} - \frac{35}{25}} = \frac{1}{5} \pm \mathrm{i}\frac{1}{5}\sqrt{34}.$$

Thus, this is a complex root, so the solution is:

$$y(x) = \boxed{c_1|x|^{1/5}\cos\left(\frac{\sqrt{34}}{5}\log|x|\right) + c_2|x|^{1/5}\sin\left(\frac{\sqrt{34}}{5}\log|x|\right)}.$$

25. Let a differential equation be defined as:

$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$

Use Euler's Method with step size $h = 1$ to approximate $y(5)$.

Solution:
With $y(0) = 0$, we have $y'(0) = 0 - 0 = 0$. We do the following steps:

- We approximate $y(1) \approx y(0) + 1 \cdot y'(0) = 0$, then we have $y'(1) \approx 1 - 0 = 1$.
- We approximate $y(2) \approx y(1) + 1 \cdot y'(1) \approx 1$, then we have $y'(2) \approx 2 - 1 = 1$.
- We approximate $y(3) \approx y(2) + 1 \cdot y'(2) \approx 2$, then we have $y'(3) \approx 3 - 2 = 1$.
- We approximate $y(4) \approx y(3) + 1 \cdot y'(3) \approx 3$, then we have $y'(4) \approx 4 - 3 = 1$.
- We approximate $y(5) \approx y(4) + 1 \cdot y'(4) \approx 4$.

Then, we have approximated that:

$$y(5) \approx \boxed{4}.$$