Final Review Set

Differential Equations

Fall 2024

Instructions:

The set of questions serves as PILOT practices to final for the Fall 2024 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- This set of problems includes questions among various topics that help students to prepare for the final exam. While this practice attempts to contain most of the materials, it might not be exhaustive in all possible topics that could appear on the final exam.
- Questions marked with asterisk(s) are more challenging, but still accessible, compared to the others.
- The questions are designed to be done without notes and calculators.

1. Find the general solution for $y = y(t)$:
$$y' + 3y = t + e^{-2t},$$
then, describe the behavior of the solution as $t \to \infty$.

2. Given an initial value problem:
$$\begin{cases} \dfrac{dy}{dt} - \dfrac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$
   (a) Find the integrating factor $\mu(t)$.
   (b) Solve for the particular solution for the initial value problem.
   (c) Discuss the behavior of the solution as $t \to \infty$ for different cases of y_0.

3. Suppose $f(x)$ is non-zero, let an initial value problem be:
$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$
   (a) Show that the differential equation is **not** linear.

   For the next two questions, suppose $f(x) = \tan x$.

(b) State, without justification, the open interval(s) in which $f(x)$ is continuous.

(c)* Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

(d)** Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

4. An autonomous differential equation is given as follows:
$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \text{ where } t \geq 0 \text{ and } y \geq 0.$$
Draw a phase portrait and sketch a few solutions with different initial conditions.

5.* Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:
$$y'(x) = e^{2x} + y(x) - 1.$$

6. Solve the following second order differential equations for $y = y(x)$:

(a) $$y'' + y' - 132y = 0.$$

(b) $$y'' - 4y' = -4y.$$

(c) $$y'' - 2y' + 3y = 0.$$

7. Given a differential equation for $y = y(t)$ being:
$$t^3y'' + ty' - y = 0.$$

(a) Verify that $y_1(t) = t$ is a solution to the differential equation.

(b)* Find the full set of solutions using reduction of order.

(c) Show that the set of solutions from part (b) is linearly independent.

8.** Given the following second order initial value problem:
$$\begin{cases} \dfrac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \\ \dfrac{dy}{dx}(1) = 0. \end{cases}$$
Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
Hint: Consider the interval in which the solution is unique.

9. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

(a) $$y'' + 2y' + y = e^{-t}.$$

(b) $$y'' + y = \tan t.$$

10. Solve for the general solution to the following higher order ODE.

(a)
$$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

(b)**
$$\frac{d^4y}{dx^4} + y = 0.$$

Hint: Consider the 8-th root of unity, *i.e.*, ζ_8, and verify which roots satisfies the polynomial.

11. Let a third order differential equation be as follows:
$$\ell\big[y(t)\big] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$
Let $\ell\big[y(t)\big] = 0$ be trivial initially.

(a) Find the set of all linearly independent solutions.

Then, assume that $\ell\big[y(t)\big]$ is non-trivial.

(b) Find the particular solution to $\ell\big[y(t)\big] = \sin t$.

(c) Find the particular solution to $\ell\big[y(t)\big] = e^{-t}$.

(d)* Suppose that $\ell\big[y_1(t)\big] = f(t)$ and $\ell\big[y_2(t)\big] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell\big[y_3(t)\big] = f(t) + g(t)$.

12. Let a system of differential equations be defined as follows, find the general solutions to the equation:

(a)
$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^2,$$

(b)*
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^3.$$

13. Solve the following initial value problem for the system of equations:
$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

14. Let a system of differential equations of $x_i(t)$ be as follows:
$$\begin{cases} x_1' = 3x_1 + 2x_2, & x_1(1) = 0, \\ x_2' = x_1 + 4x_2, & x_2(1) = 2. \end{cases}$$

(a) Solve for the solution to the initial value problem.

(b) Identify and describe the stability at equilibrium(s).

15. Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle θ (counter-clockwise), then:
$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a)* Show that $M^{\intercal} = M^{-1}$.

(b)** Let $\theta = 2\pi/k$ be fixed, where k is an integer. Find the least positive integer n such that

$M^n = \mathrm{Id}_2$. Here, n is called the *order* of M.
Hint: Consider the rotational matrix geometrically, rather than arithmetically.

16. Let a non-linear system be:
$$\frac{dx}{dt} = x - y^2 \text{ and } \frac{dy}{dt} = x + x^2 - 2y.$$
Verify that $(0,0)$ is a critical point and classify its type and stability.

17. Let a system of non-linear differential equations be defined as follows:
$$\begin{cases} x' = 2x + 3y^2, \\ y' = x + 4y^2. \end{cases}$$
Find all equilibrium(s) and classify their stability locally.

18. Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:
$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$
Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.
   (a) Find the set of all equilibrium(s) for $\mathbf{x}$.
   (b) Find the set in which the equilibrium(s) is locally linear.

   Now, $F : \mathbb{R}^2 \to \mathbb{R}$ is not necessarily well-behaved.
   (c)** Construct a function F such that $\mathbf{x}$ has a equilibrium that is <u>not</u> locally linear.
   Hint: Consider the condition in which a non-linear system is locally linear.

19. Let a system of (x, y) be functions of variable t, and they have the following relationship:
$$x' = (1 + x)\sin y \text{ and } y' = 1 - x - \cos y.$$
   (a) Identify the corresponding linear system.
   (b) Evaluate the stability for the equilibrium at $(0,0)$ by showing it is locally linear.

20.** Let a locally linearly system be defined as:
$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x} + \mathbf{f}(\mathbf{x}),$$
where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function. Find the necessary condition(s) in which the equilibrium(s) have a stable *center* in linear system. Then, state the stability and type (if possible).
Hint: Consider the solution for the linear case or matrix exponential.

21. Given the a system of differential equations as follows:
$$\begin{cases} x' = x - y - x(x^2 + y^2), \\ y' = x + y - y(x^2 + y^2). \end{cases}$$
Find the limit cycle of the system, classify the critical points, and sketch a phase portrait of the system.

22. Consider the following series. Identify if such series converges. Compute the limit if the series converges.

(a)
$$\sum_{k=0}^{\infty} \frac{n!}{2^n}.$$

(b)*
$$\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!}.$$

(c)
$$\sum_{k=0}^{\infty} \frac{x^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!}.$$

23. Use the *series expansions* to find the solutions to the following differential equation:
$$y'' + 3y' = 0.$$

24. Use the *Euler's equation* to find the solution to the following differential equations:

(a)
$$x^2y'' + 5xy' + 4y = 0.$$

(b)
$$5x^2y'' + 3xy' + 7y = 0.$$

25. Let a differential equation be defined as:
$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$
Use Euler's Method with step size $h = 1$ to approximate $y(5)$.

Congratulations for completing the final review problem set.
This marks the end of PILOT learning for Differential Equations in the Fall 2024 term.

We hope that you have consolidated your knowledge, learned more concepts, and better understood the materials through the semester. We wish the best of luck for your final examination as well as the rest of your academic career.

Sincerely,
PILOT for Differential Equations.