PILOT Final Exam Review

Differential Equations

Johns Hopkins University

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As you prepare for Final exam, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/FA25/`
  - Find all the review problem sets.
    - Please remember all problem sets since the final is cumulative.
  - Consult the archives page for PILOT sets from the semester.
- Review the *homework/quiz sets* provided by the instructor.
- Join the PILOT Final Review Session. (You are here.)

Part 1:
Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

1 Preliminaries

2 First Order ODEs

3 Second Order ODEs

4 Higher Order ODEs

5 System of First Order Linear ODEs

6 Non-linear Systems

7 Laplace Transformation

8 Series Solutions to Second-Order Linear Equations

9 Numerical Methods

Preliminaries

- Classifications of Differential Equations
  - ODEs vs PDEs
- Modeling Using ODEs
  - Half Life Problem

When having various differential equations, we can classify them by their properties.

ODEs vs PDEs

Ordinary Differential Equations (ODEs) involves ordinary derivatives ($\frac{dy}{dt}$), while Partial Differential Equations (PDEs) involves partial derivatives ($\frac{\partial y}{\partial t}$).

This course focuses on ODEs, and it can also be classified in various different ways:

- **Single equation** involves one unknown and one equation, while **system of equations** involves multiple unknowns and multiple equations.
- The **order** of the differential equation is the order of the highest derivatives term.
- **Linear** differential equations has only linear dependent on the function, while **non-linear** differential equations has non-linear dependent on the function.

ODEs can be used for modeling. During modeling, it often follows the following steps:

1. Construction of the Models,
2. Analysis of the Models,
3. Comparison of the Models with Reality.

An example of modeling is the **half-life problem**.

Half Life Problem

The physics model for half life indicates the relationship between half life (τ) of a substance of amount $N(t)$ with initial amount N_0 at a time t is:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{\tau}},$$

where the rate of decay (λ) and half life (τ) are related by:

$$\tau \times \lambda = \log 2.$$

First Order ODEs

- Methods of Solving ODEs
  - Separable ODEs
  - Integrating Factor
- Existence and Uniqueness Theorems
- Autonomous ODEs
  - Rational Root Test
- Logistic Population Growth
  - Partial Fractions
- Exactness Problem
  - Integrating Factor for Non-Exact Case
- Bifurcation
  - Bifurcation Diagram

Here, we will introduce various ways of solving ODEs:

Separable ODEs

For ODEs in form $M(t) + N(y)\frac{dy}{dt} = 0$, it can be separated by:

$$M(t)dt + N(y)dy = 0.$$

When the ODE is not separable, we may consider using the **integrating factor**.

Integrating Factor

For ODEs in form $\frac{dy}{dt} + a(t)y = b(t)$, the integrating factor is:

$$\mu(t) = \exp\left(\int a(t)dt\right).$$

The existence and uniqueness for Initial Value Problem (IVP) tells us information on if the we can obtain a unique solution over some interval:

- For an IVP in simple form:

$$\begin{cases} \dfrac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$

  If $a(t)$ and $b(t)$ are continuous on an interval (α, β) and $t_0 \in (\alpha, \beta)$. Then, there exists a uniqueness solution y for (α, β) to the IVP.

Picard's Theorem

- For an IVP in general form:

$$\begin{cases} \dfrac{dy}{dt} = f(t,y), \\ y(t_0) = y_0. \end{cases}$$

For $t_0 \in I = (a,b)$, $y_0 \in J = (c,d)$, if $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous on interval $I \times J$. Then, there exists a unique solution on a smaller interval $I' \times J' \subset I \times J$, in which $(t_0, y_0) \in I' \times J'$.

Only Contrapositive is Guaranteed to be True

For both theorems, you can conclude that if *there does not exist a solution or the solution is not unique*, then *the conditions must not be satisfied*. You **cannot** conclude that if *the conditions are not satisfied*, then *there is no unique solution*.

Autonomous ODEs are in form of:

$$\frac{dy}{dt} = f(y).$$

The stability (stable/semi-stable/unstable) of equilibrium can be determined by phase lines, *i.e.*, the zeros of the function $f(t)$.

Rational Root Test

Let the polynomial with integer coefficients be defined as:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0,$$

then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p | a_0$ and $q | a_n$.

The logistic population growth model with population (y), growing rate (r), and carrying capacity (k) is given by:

$$\begin{cases} \dfrac{dy}{dt} = r\left(1 - \dfrac{y}{k}\right) y, \\ y(0) = y_0, \end{cases}$$

whose general solution is $y(t) = \dfrac{ky_0}{(k - y_0)e^{-rt} + y_0}$.

Partial Fractions

For a fraction in the form $\dfrac{C}{(x - a_1)^{n_1}(x - a_2)^{n_2} \cdots (x - a_m)^{n_m}}$, it can be decomposed in terms of:

$$\frac{C_{1,1}}{x - a_1} + \frac{C_{1,2}}{(x - a_1)^2} + \cdots + \frac{C_{1,n_1}}{(x - a_1)^{n_1}} + \cdots + \frac{C_{m,1}}{x - a_m} + \cdots + \frac{C_{m,n_m}}{(x - a_m)^{n_m}}.$$

The condition for a function in form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ to be exact is:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

For solving Exact ODEs, either finding $\int M(x,y)dx + h(y)$ or $\int N(x,y)dy + h(x)$ and taking partials again to fit gives the solution $\Psi(x,y) = C$.

Integrating Factor for Non-Exact Case

$$\mu(t) = \exp\left(\int \frac{M_y - N_x}{N} dx\right) \quad \text{or} \quad \mu(t) = \exp\left(\int \frac{N_x - M_y}{M} dy\right).$$

When a differential equation contains some unknown, fixed parameter C, its equilibriums would exhibit different behavior, the bifurcation value is the critical value such that the equilibriums have different stability.

Bifurcation Diagram

A bifurcation diagram is the vertical concatenation of phase portraits (C-y plot), in which the equilibriums will be marked for respective values of C.

Second Order ODEs

- Linear Homogeneous Cases
  - Complex Characteristic Roots
  - Repeated Characteristic Roots
- Linear Independence
  - Definition of Linearly Independence
  - Superposition Principle
- Reduction of Order
  - Product Rule and Chain Rule
- Non-homogeneous Cases
  - Variation of Parameters
  - Undetermined Coefficients

Consider the linear homogeneous ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is $r^2 + pr + q = 0$, with real, distinct solutions r_1 and r_2, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t} \cos(\beta t) + c_2 e^{\lambda t} \sin(\beta t).$$

Repeated Characteristic Roots

If the solutions are repeated, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

To form a fundamental set of solutions, the solutions need to be linearly independent, in which the Wronskian (W) must be non-zero, meaning that:

$$\mathrm{W}[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \cdots, f_n, \cdots\}$ is linearly independent when for $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots \in \mathbb{F}$ (typically $\mathbb{C}$):

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n + \cdots = 0 \iff \lambda_1 = \lambda_2 = \cdots = \lambda_n = \cdots = 0.$$

Superposition Principle

If $y_1(t)$ and $y_2(t)$ are solutions to $l[y] = 0$, then the solution $c_1 y_1(t) + c_2 y_2(t)$ are also solutions for all constants $c_1, c_2 \in \mathbb{R}$.

For non-linear second order homogeneous ODEs, when one solution $y_1(t)$ is given, the other solution is in form:

$$y_2(t) = u(t) \cdot y_1(t).$$

Product Rule and Chain Rule

- **Product Rule**: $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x)g(x) + f(x)\frac{dg}{dx}(x).$
- **Chain Rule**: $\frac{d}{dx}[f(g(x))] = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).$

Procedure of Reduction of Order

As long as $y_1(t)$ is a solution, you will be able to reduce the differential equation with respect to y_2 into a differential equation involving only $u''(t)$ and $u'(t)$ terms to solve for $\omega(t) = u'(t)$.

Let the differential equation be:

$$Ay''(t) + By'(t) + Cy(t) = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$ and $y_2(t)$ be the two homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

Variation of Parameters

The particular solution of the differential equation can be written as the integrals of respective parts.

$$y_p = y_1(t) \int \frac{-y_2(t) \cdot g(t)}{W} dt + y_2(t) \int \frac{y_1(t) \cdot g(t)}{W} dt.$$

Another approach is less calculation intensive, but requires the function $g(t)$ to be constrained in certain forms.

Undetermined Coefficients

A guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$	**Guess**
Polynomials: $\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.: $\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \cos(ax)$
Exp.: e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

Higher Order ODEs

- Existence and Uniqueness Theorem
- Homogeneous Cases
  - Complex Characteristic Roots
  - Repeated Characteristic Roots
- Linear Independence
  - Definition of Linearly Independence
- Abel's Formula
- Non-Homogeneous Cases
  - Variation of Parameters
  - Undetermined Coefficients

For higher order IVP in form:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots,y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

If $P_0(t), P_1(t), \cdots, P_{n-1}(t)$, and $g(t)$ are continuous on an interval I containing t_0. Then there exists a unique solution for $y(t)$ on I.

Only Contrapositive is Guaranteed to be True

Again, for this theorem, you can conclude that if *there does not exist a solution or the solution is not unique*, then *the conditions must not be satisfied*. You **cannot** conclude that if *the conditions are not satisfied*, then *there is no unique solution*.

The higher order homogeneous ODEs are in form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

By computing the characteristic equation $r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$, with solutions $r_1, r_2, \cdots, r_n$, the general solution is $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$.

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t}\cos(\beta t) + c_2 e^{\lambda t}\sin(\beta t) + \text{ rest of the solutions.}$$

Repeated Characteristic Roots

If the solutions are repeated with multiplicity m, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} + \cdots c_m t^{m-1} e^{rt} + \text{ rest of the solutions.}$$

To obtain the fundamental set of solutions, the Wronskian (W) must be non-zero, where Wronskian is:

$$\mathrm{W}[y_1, y_2, \cdots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \cdots, f_n, \cdots\}$ is linearly independent when for $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots \in \mathbb{F}$ (typically $\mathbb{C}$):

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n + \cdots = 0 \iff \lambda_1 = \lambda_2 = \cdots = \lambda_n = \cdots = 0.$$

For higher order ODEs in the form of:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots,\ y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

Its Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = Ce^{\int -P_{n-1}(t)dt},$$

where C is independent of t but depend on $y_1, y_2, \cdots, y_n$.

Let the differential equation be:

$$L[y^{(n)}(t), y^{(n-1)}(t), \cdots, y(t)] = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$, $y_2(t)$, $\cdots$, $y_n(t)$ be all homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

Variation of Parameters

The particular solution is:

$$y_p = y_1(t) \int \frac{W_1 g}{W} dt + y_2(t) \int \frac{W_2 g}{W} dt + \cdots + y_n(t) \int \frac{W_n g}{W} dt,$$

where W_i is defined to be the Wronskian with the i-th column alternated into $\begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^\intercal$.

Undetermined Coefficients

Same as in degree 2, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$	**Guess**
Polynomials: $\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.: $\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \cos(ax)$
Exp.: e^{at}	Ce^{at}

Again, the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

System of First Order Linear ODEs

- Solving for Eigenvalues and Eigenvectors
- Linear Independence
  - Abel's Formula
- Phase Portraits
  - Node Graph
- Repeated Eigenvalues
  - Algebraic Multiplicity and Geometric Multiplicity
- Phase Portraits
  - Node Graph
  - Spiral/Center Graph
  - Repeated Eigenvalue Graph

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - rI) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - rI) \cdot \boldsymbol{\xi} = \mathbf{0}.$$

Suppose that the eigenvalues are distinct and the eigenvectors are linearly independent, the solution to the ODE is:

$$\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t} + \cdots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}.$$

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det\left(\Psi(t)\right).$$

The system is linearly independent if the Wronskian is non-zero.

Abel's Formula

For the linear system in form:

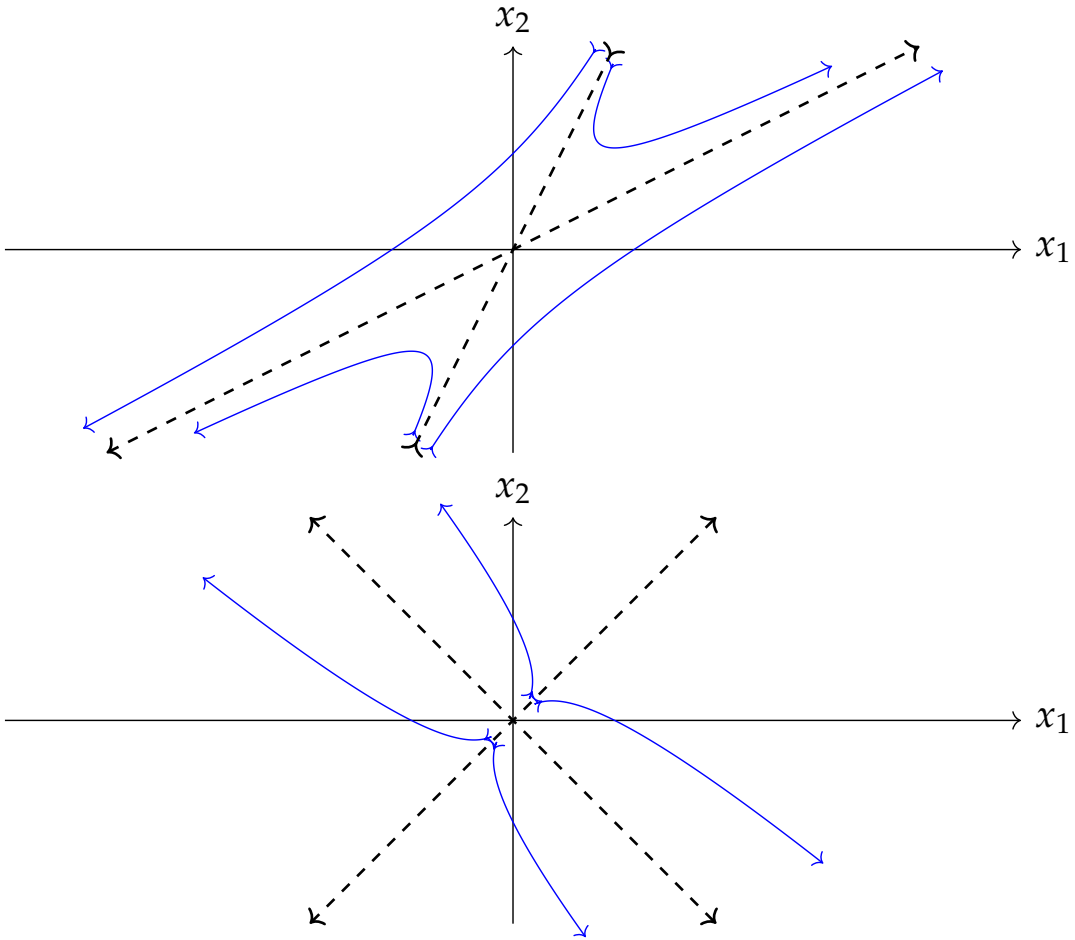
$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A, which is the sum of the diagonals, that is:

$$W = Ce^{\int \text{trace}\, A dt} = Ce^{\int (A_{1,1}+A_{2,2}+\cdots+A_{n,n})dt}.$$

In particular, we can sketch the linear system of $\mathbb{R}^2$ in terms of phase portraits given the eigenvalues and eigenvectors.

- For a node graph, we have it as (directions might vary):



For repeated eigenvalue r with only one (linearly independent) eigenvector, if a given a solution is $\mathbf{x}^{(1)} = \boldsymbol{\xi} e^{rt}$, the other solution would be:

$$\mathbf{x}^{(2)} = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt},$$

where $(A - Ir) \cdot \boldsymbol{\eta} = \boldsymbol{\xi}$, and $\mathbf{x}^{(2)}$ is called the *generalized eigenvector*.

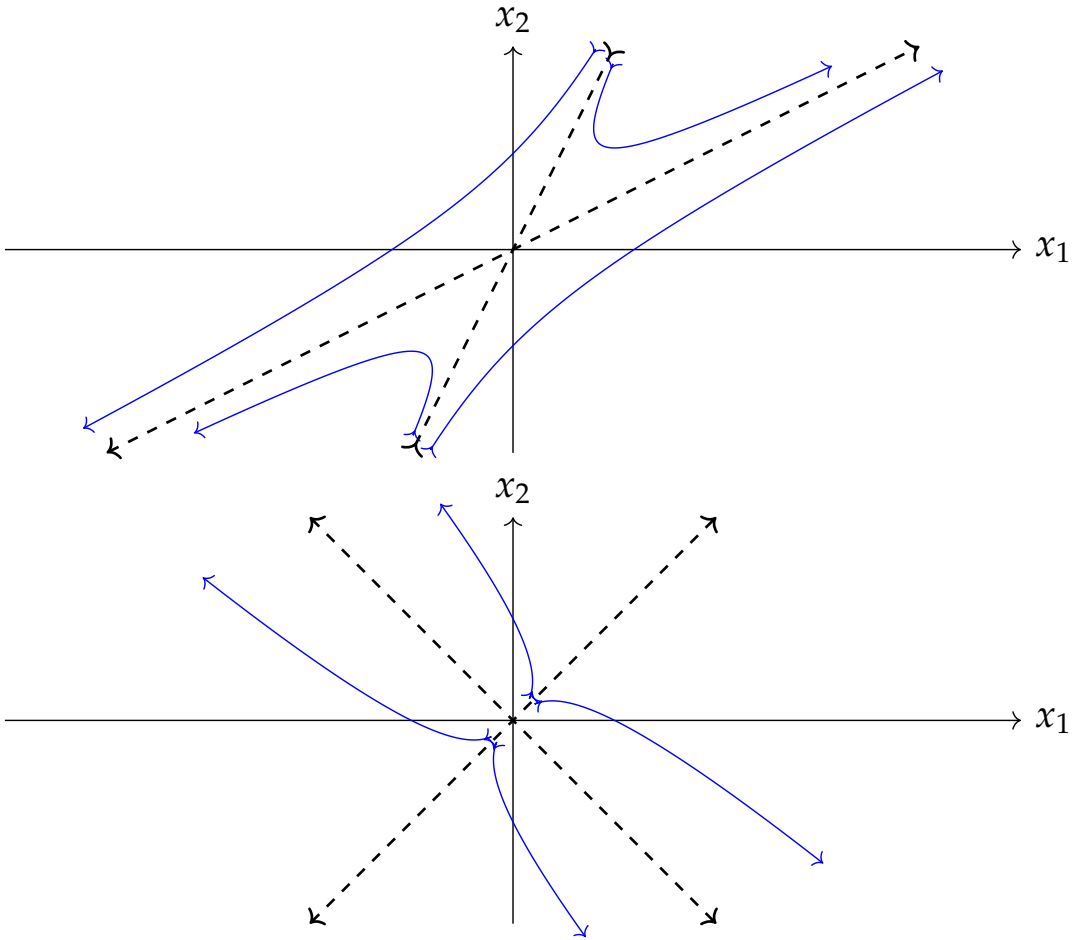
Algebraic Multiplicity and Geometric Multiplicity

The algebraic multiplicity refers to the multiplicity of root in the characteristic polynomial, and the geometric multiplicity refers to the dimension of the eigenspace associated with the eigenvalue.

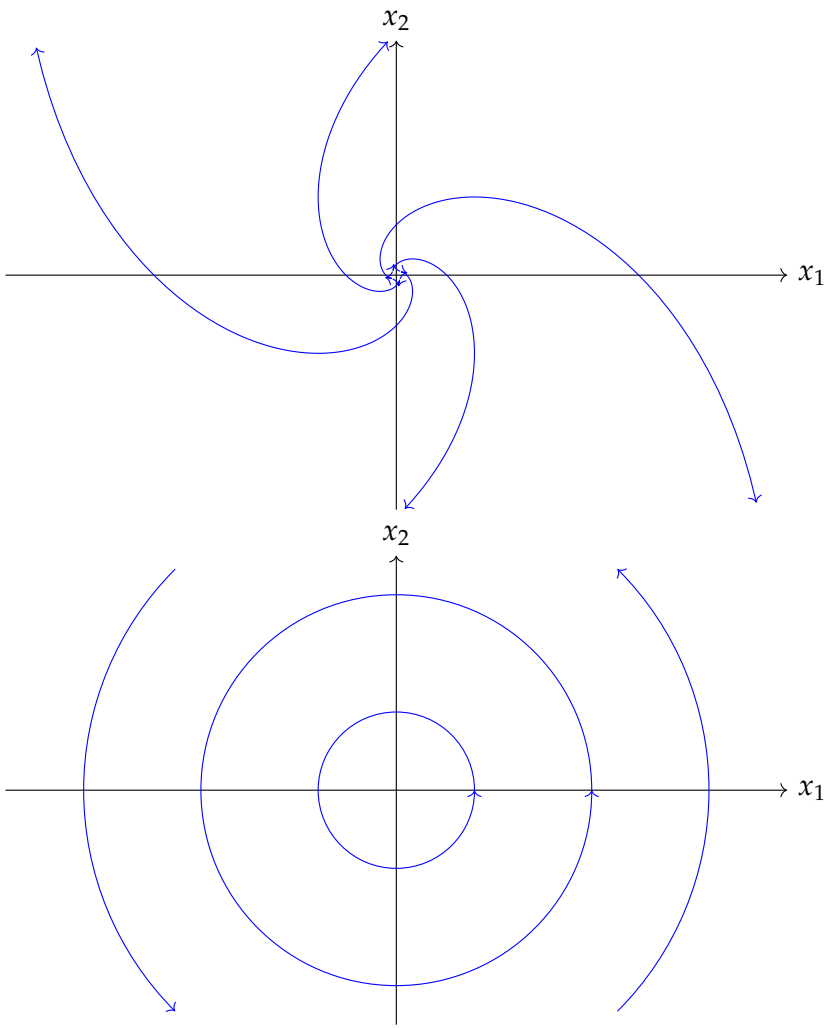
- The algebraic multiplicity will be no less than the geometric multiplicity for each eigenvalue.
- We need the generalized eigenvector when the algebraic multiplicity is larger than the geometric multiplicity.

In particular, we can sketch the linear system of $\mathbb{R}^2$ in terms of phase portraits given the eigenvalues and eigenvectors.

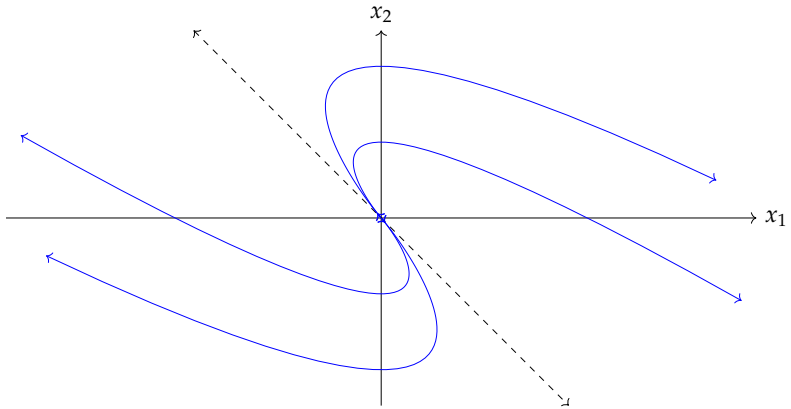
- For a node graph, we have it as (directions might vary):



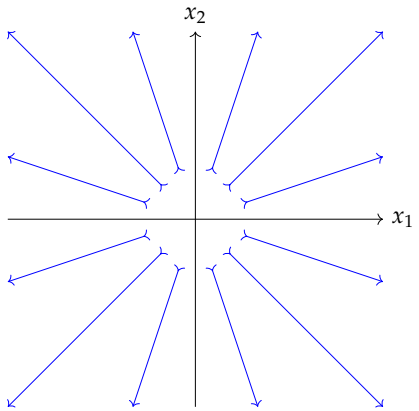
- For a spiral/center graph, we have it as (directions might vary):



- For repeated eigenvalues with less geometric multiplicity, the solution is (directions might vary):



- If the geometric multiplicity is the same, the graph is simply a radial shape (directions might vary):



Non-linear Systems

- Linear Approximation
  - Autonomous Systems
- Stability
- Limit Cycles

For non-linear system $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}'$, if $F, G \in C^2$ and the system is locally linear, the approximation at critical point (x_0, y_0) is:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \mathbf{J}(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where Jacobian is:

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}.$$

Autonomous Systems

When $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} F(y) \\ G(x) \end{pmatrix}$, it can be solved implicitly for:

$$\frac{dy}{dx} = \frac{G(x)}{F(y)}.$$

For linearized system with eigenvalues r_1, r_2, the stability can be concluded as follows:

Eigenvalues	**Linear System**		**Nonlinear System**	
	Type	**Stability**	**Type**	**Stability**
Eigenvalues are λ_1 and λ_2				
$0 < \lambda_1 < \lambda_2$	Node	Unstable	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable	Node or Spiral Point	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable	Node or Spiral Points	Asymptotically Stable
Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$				
$\alpha > 0$	Spiral Point	Unstable	Spiral Point	Unstable
$\alpha = 0$	Center	Stable	Center or Spiral Point	Indeterminate
$\alpha < 0$	Spiral Point	Asymptotically Stable	Spiral Point	Asymptotically Stable

A closed trajectory or periodic solution repeats back to itself with period τ:

$$\begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Closed trajectories with either side converging to/diverging from the solution is a limit cycle.

Conversion to Polar Coordinates

A Cartesian coordinate can be converted by:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ rr' = xx' + yy', \\ r^2\theta' = xy' - yx'. \end{cases}$$

For a linear system $x = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$ with $F, G \in C^1$:

1. A closed trajectory of the system must enclose at least 1 critical point.
2. If it only encloses 1 critical point, then that critical point cannot be saddle point.
3. If there are no critical points, there are no closed trajectories.
4. If the unique critical point is saddle, there are no trajectories.
5. For a simple connected domain D in the xy-plane with no holes. If $F_x + G_y$ had the same sign throughout D, then there is no closed trajectories in D.

Laplace Transformation

- Properties of Laplace Transformation
- Properties of Laplace Transformation:
- Elementary Laplace Transformations
- Step Functions:
- Impulse Functions
- Convolution

The Laplace Transformation of a function f is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

Note that Laplace Transformation can be used on non-continuous functions by utilizing step functions.

Laplace Transformation has the following properties:

1. Laplace Transformation is a linear operator:
$$\mathcal{L}\{f + \lambda g\} = \mathcal{L}\{f\} + \lambda\mathcal{L}\{g\}$$
2. Laplace Transformation for derivatives:
$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0),$$
$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0),$$
$$\vdots$$
$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0).$$
3. First Shifting Theorem:
$$\mathcal{L}\{e^{ct}f(t)\} = F(s - c).$$

The Laplace Transformations can be used for solving IVP, where the inverse helps to find the original function prior to transformation.

The Laplace Transformations for elementary functions are given in the following table, note that they can still be calculated by its definition:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$t^n, n \in \mathbb{Z}_{>0}$	$\frac{n!}{s^{n+1}}, s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}, s > 0$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$

The step functions are defined by:

$$u_c(t) = u(t-c) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

And the Laplace Transformations of the step function is:

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}.$$

The step function forms the Second Shifting Theorem:

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s).$$

The idealized unit impulse function $\delta(t)$, or *Dirac delta function*, satisfies the properties that:

$$\delta(t) = 0 \text{ for } t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

There is no ordinary function satisfying the idealized unit impulse function, so it is a generalized function.
A unit impulse at an arbitrary point $t = t_0$, denoted by $\delta(t - t_0)$, follows that:

$$\delta(t) = 0 \text{ for } t \neq t_0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t - t_0)dt = 1.$$

The Laplace Transformation of the impulse function is:

$$\mathcal{L}\{\delta(t - c)\} = e^{-cs}.$$

The convolution of f and g, denoted $(f * g)$, is defined as:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau.$$

The convolution $f * g$ has many of the properties of ordinary multiplication:

1. Commutativity: $f * g = g * f$;
2. Distributivity: $f * (g + h) = f * g + f * h$;
3. Associativity: $(f * g) * h = f * (g * h)$;
4. Zero Property: $f * 0 = 0 * f = 0$, where 0 is a function that maps any input to 0.

The Laplace Transformation of the convolution of f and g is:

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s).$$

Series Solutions to Second-Order Linear Equations

- Power Series
- Ordinary Point
- Euler Equations
- Regular Singular Point

A power series is an infinite series in the form:

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots,$$

where a_n is the coefficient for term n and c is the center of the approximation.

A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *converge* at a point x if:

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-x_0)^n \text{ exists for that } x.$$

A power series *converges pointwise* on X if it converges on every $x \in X$.

A power series converges absolutely at a point x if the power series:

$$\sum_{n=0}^{\infty} |a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||x-x_0|^n \text{ converges.}$$

Note that absolute converges implies convergence, but the converse is not true.

Here are some properties of series:

1. (Ratio test). If $a_n \neq 0$, and if for a fixed value of x, and:
$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{x_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L,$$
then the power series converges absolutely at x if $|x-x_0|L<1$ and diverges if $|x-x_0|L>1$.
2. (Monotonic property). If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at $x=x_1$, then it converges absolutely for $|x-x_0|<|x_1-x_0|$. If it diverges at $x=x_1$, then it diverges for $|x-x_0|>|x_1-x_0|$.
3. (Radius of convergence). Let $\rho>0$ be such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for $|x-x_0|<\rho$ and diverges for $|x-x_0|>\rho$, then ρ is the *radius of convergence* and $(x_0-\rho, x_0+\rho)$ is the *interval of convergence*.

Also, we note that power series can be added or subtracted term-wise. They can also be multiplied and divided by having divisions of terms.
Recall that by Taylor theorem, suppose $f \in C^\infty$, then we can form the Taylor polynomial as a power series, with coefficient:

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

In particular, if f has a Taylor polynomial at x_0 with a positive radius of convergence, we say the series is *analytic* at x_0.

Here, we are thinking of the second order homogeneous differential equation, namely:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0.$$

Additionally, we suppose that P, Q, and R are polynomials and have no factor common factor $(x - c)$. Thus, we have $P(x_0) \neq 0$ being an ordinary point. When $P(x_0) = 0$, it is a singular point (or pole).

When we generalize, we will have:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

where p and q are any functions. Similarly, consider x_0 where both p and q are analytic, x_0 is *ordinary*, otherwise, it is *singular*. Here, we say $p(x)$ has singularity of a pole at x_0 of order n if:

$$(x - x_0)^n p(x) \text{ is analytic at } x_0.$$

Assuming absolute convergence, one can apply the derivative operator on the sequence, that is:

$$\begin{aligned}\frac{d}{dx}\left[\lim_{N\to\infty}\sum_{n=0}^{N} a_n(x-x_0)^n\right] &= \lim_{N\to\infty}\left[\frac{d}{dx}\sum_{n=0}^{N} a_n(x-x_0)^n\right] \\ &= \lim_{N\to\infty}\sum_{n=1}^{N} a_n n(x-x_0)^{n-1}.\end{aligned}$$

Often, we we apply the derivative operator, we will notice some *recurrence relation*, that is the successive coefficients can be evaluated one by one.
In particular, when we have a power series:

$$\varphi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

by taking the m-th derivative and evaluating it at 0, we will have:

$$\frac{d^m \varphi}{dx^m}(x_0) = m! a_m.$$

In the section, we go back to the focus of:

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

where P, Q, R are polynomials with no common factors.
For the Euler's equation, we consider the differential equation in the form:

$$x^2y'' + \alpha xy' + \beta y = 0.$$

Then, $|x|^r$ is a solution to the above differential equation if r is a solution to $r(r-1) + \alpha r + \beta = 0$.

Let r_1, r_2 be the roots of $r(r-1) + \alpha r + \beta = 0$, then the solution to the differential equation can be represented by:

- When $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then:
$$y(x) = c_1|x|^{r_1} + c_2|x|^{r_2}.$$
- When $r_1, r_2 \in \mathbb{R}$ and $r := r_1 = r_2$, then:
$$y(x) = c_1|x|^{r} + c_2 \log|x| \cdot |x|^{r}.$$
- When $r_1, r_2 = \lambda + i\mu \in \mathbb{C}$ and $\mu \neq 0$, then:
$$y(x) = c_1|x|^{\lambda} \cos(\mu \log|x|) + c_2|x|^{\lambda} \sin(\mu \log|x|).$$

Now, we want to research on the case when x_0 is a regular singular point, that is for equation:

$$y'' + p(x)y' + q(x)y = 0,$$

and x_0 satisfies that:

1. x_0 is a singular point, and
2. $p(x)$ has a pole of order 1 and $q(x)$ has a pole of order no more than 2.

A singular point that is not regular is a irregular singular point.

Without loss of generality, we may horizontally shift the equation to obtain that $x = 0$ is a regular singular point. Then, we may write:

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \qquad \text{and} \qquad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$$

on some interval $|x| < \rho$ within the radius of convergence.

Hence, we may multiply x^2 on both side, giving us that:

$$x^2y'' + x\underbrace{\big(xp(x)\big)}_{\tilde{p(x)}}y' + \underbrace{\big(x^2q(x)\big)}_{\tilde{q}(x)}y = 0,$$

in which $\tilde{p}$ and $\tilde{q}$ are analytic at $x = 0$. Then, we will be able to Euler Equations to solve for the differential equation with respect to $\tilde{p}$ and $\tilde{q}$.

Numerical Methods

- Euler's Method
- Generalization on Euler's Method

The numerical approximation focuses on first-order initial value problem:

$$\begin{cases} \dfrac{dy}{dt} = f(t, y), \\ y(t_0) = y_0. \end{cases}$$

By the *Existence and Uniqueness Theorem*, a unique solution exists for some rectangular region containing (t_0, y_0) when f and $\frac{\partial f}{\partial y}$ are continuous. With this foundation, we may apply Euler's method on such region. (*Note that* out of the region, the approximation would not be accurate.)

Euler's method recursively applies the following function:

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n), \quad n = 0, 1, 2, \cdots,$$

and when the steps are constrained to be a constant h, we have:

$$y_{n+1} = y_n + hf(t_n, y_n), \quad n = 0, 1, 2, \cdots.$$

Typically, Euler's method incurs error, whereas some typical issues are:

1. When the step size h is too big, the error is significant.
2. When the step size h is too small, the cost of calculation is expensive.
3. The computation does address the asymptotic behaviors.
4. When the vector field has steep components, the approximation differs more.

Euler's method can be analyzed by using the *Fundamental Theorem of Calculus*, that is:

$$\begin{aligned} y(t) &= y(t_n) + \int_{t_n}^{t} f\big(s, y(s)\big)ds \\ &\approx y(t_n) + \sum_{t_0 \le t_i < t_{i+1} \le t} f(t_i, y_n)(t_{i+1} - t_i), \end{aligned}$$

in which we may establish the improved Euler's Method, by:

$$y_{n+1} = y_n + h\left(\frac{f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))}{2}\right),$$

by considering the trapezoid approach for *Riemann sum*. Since the $f(t, y)$ depends only on t and not on y, then solving differential equation reduced from $y' = f(t, y)$ to integrating $f(t)$, which makes the improved Euler's Method into:

$$y_{n+1} = y_n + \frac{h}{2}\big(f(t_n) + f(t_n + h)\big).$$

Part 2:
Open Poll

We will work out some sample questions.

- Please let us know if you have questions from the Review Problem Sets or Weekly Problem Sets.
- Let us know if you want to go through any concepts that you are not sure with.

Thank you for being with **Differential Equations PILOT** this semester.

Good luck on your finals, and best wished for your future pursuits!

While you prepare for the final:
Collaborate with your friends and classmates to study together!
Get physically and mentally prepared. Get enough sleep and some food before the exam!
Believe in yourself. You are amazing and you can make it this!