Additional Material: Algebraic and Geometric Multiplicity

Differential Equations

Fall 2025

When you find eigenvalues with multiplicity, we might not necessarily find as many linearly independent eigenvectors while doing the computation. Hence, we will be investigating what are with these eigenvalues and eigenvectors in this additional exercise.

Definition. (Algebraic and Geometric Multiplicity). We define the *algebraic multiplicity* of an eigenvalue as its multiplicity as a root to the characteristic polynomial, and the *geometric multiplicity* is the dimension of the eigenspace.

This problem investigates the case for repeated eigenvalues. First, we let the matrix $A \in \mathbb{R}^{2\times 2}$ be:

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

(a) Find the eigenvalue and its corresponding eigenvector. State their multiplicities.

(b) Find a the general solution to $\mathbf{x}' = A \cdot \mathbf{x}$, where $\mathbf{x} = (x_1, x_2)$.

Then, we consider the diagonal n-by-n matrices, that is matrices with entries only on the diagonal, which can be characterized as:

$$D = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

(c) Show that the eigenvalues are exactly $a_1, \cdots, a_n$, and the algebraic multiplicity is exactly the same as geometric multiplicity for all eigenvalues.

(d) Consider the linear system $\mathbf{x}' = D \cdot \mathbf{x}$ for $\mathbf{x} = (x_1, \cdots, x_n)$. Explain why do not have to find the eigenvalues in this case.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) For the eigenvalue and eigenvector, we set:

$$0 = \det \begin{pmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = (2-\lambda)(-\lambda)+1 = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2.$$

Hence, the eigenvalue is $\boxed{\text{1 with algebraic multiplicity 2}}$.

Then, we consider the eigenvector, that is $\boldsymbol{\xi}$ such that $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.\boldsymbol{\xi} = \mathbf{0}$, hence $\xi_1 + \xi_2 = 0$, so we have $\xi_2 = -\xi_1$, so the eigenvalue is $\boxed{(1,-1) \text{ with geometric multiplicity 1}}$.

(b) To find the solution to the differential equations, we first obtain a solution as:

$$\mathbf{x}^{(1)}(t) = \boxed{C_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}}.$$

However, for the linear system, we want another solution with the repeated roots. Here, we think of the second root by having the vector $\boldsymbol{\eta}$ such that $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.\boldsymbol{\eta} = \boldsymbol{\xi}$, so we have:

$$\begin{cases} \eta_1 + \eta_2 = \xi_1 = 1, \\ -\eta_1 - \eta_2 = \xi_2 = -1. \end{cases}$$

This solves to $\eta_1 + \eta_2 = 1$, so we have $\boldsymbol{\eta} = (\eta_1, 1-\eta_1)$, and here, we consider $(0,1)$ since $(1,-1)$ is the eigenvector already, so the second solution is:

$$\mathbf{x}^{(2)}(t) = \boxed{C_2 \left(te^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)}.$$

(c) *Proof.* Note that as we subtract λ on the diagonal, we have:

$$D - \lambda \,\mathrm{Id} = \begin{pmatrix} a_1-\lambda & 0 & \cdots & 0 \\ 0 & a_2-\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n-\lambda \end{pmatrix}.$$

Here, by the recursive definition of determinant, we disregard the zeros in the first row and entry, we have:

$$\begin{aligned} 0 &= \det \begin{pmatrix} a_1-\lambda & 0 & \cdots & 0 \\ 0 & a_2-\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n-\lambda \end{pmatrix} = (a_1-\lambda)\det \begin{pmatrix} a_2-\lambda & 0 & \cdots & 0 \\ 0 & a_3-\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n-\lambda \end{pmatrix} \\ &= (a_1-\lambda)(a_2-\lambda)\det \begin{pmatrix} a_3-\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n-\lambda \end{pmatrix} = (a_1-\lambda)(a_2-\lambda)\cdots(a_n-\lambda). \end{aligned}$$

Therefore, we know that the eigenvalues are exactly the diagonal matrices, with algebraic multiplicity as how many times they appear. Now, when we shift to the geometric multiplicity, it becomes the *kernel* (or null space) for $D - a_i \,\mathrm{Id}$, which we have:

$$D - a_i \,\mathrm{Id} = \begin{pmatrix} a_1 - a_i & 0 & \cdots & 0 \\ 0 & a_2 - a_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n - a_i \end{pmatrix}.$$

Here, we consider the matrix as column vectors and only the columns with zero is increasing the dimension of the kernel by 1, so the dimension of the eigenspace is exactly the number of times that a_i appears, hence the geometric multiplicity is exactly the algebraic multiplicity. □

(d) With a diagonal system, our linear system degenerates into:

$$\begin{cases} x_1' = a_1 x_1, \\ x_2' = a_2 x_2, \\ \vdots \\ x_n' = a_n x_n. \end{cases}$$

Hence, the solution is exactly:

$$\boxed{\begin{cases} x_1(t) = C_1 e^{a_1 t}, \\ x_2(t) = C_2 e^{a_2 t}, \\ \vdots \\ x_n(t) = C_n e^{a_n t}. \end{cases}}$$

Note that this is really a simple case of single first order linear differential equations. *We suggest diligent readers to also think about how to solve if this is a linear system.*