Problem Set 10

Differential Equations

Fall 2025

As of right now, we are finished with the first order linear system. As we are about to conclude this chapter, let us review what we have learned about linear systems:

- Concepts:
  - Vector space
  - Eigenspace
  - Existence & Uniqueness Theorem
- Methods to solve linear systems of ODEs:
  - Eigenvectors & Eigenvalues
  - Repeated Roots
  - Phase Portraits and Stability

As we hop into non-linear systems, we will consolidate the concepts about linear systems.

Tip of the Week

1. (Complex Eigenvalue and Phase Portraits). Find the general solution and sketch a few phase portraits for the following problems:

   (a) $$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \mathbf{x}.$$

   (b) $$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{x}.$$

2. (Another "Big Guy"). Let $\mathbf{x} \in \mathbb{R}^4$, find the general solution of $\mathbf{x}$ for:

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \mathbf{x}.$$

3. (Zero Eigenvalue). Let a system of $\mathbf{x} = (x_1, x_2)$ be defined as:

$$\mathbf{x}' = \begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} \cdot \mathbf{x}.$$

(a) Find the eigenvalues and eigenvectors for $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix}$.

(b) Give a full set of solutions to the differential equation. Plot some trajectory on the x_1x_2-plane.

(c) Let A be an arbitrary square matrix. Show that A is non-invertible if and only if A has zero as an eigenvalue.

Note: Please avoid using the definition that the determinant is the product of all eigenvalues. Moreover, consider the geometric implication of eigenvalue to account for invertibility.

4. (Phase Portraits for Repeated Roots). Find the solutions to the following linear system differential equation, sketch a few phase portraits, and classify its type and stability.

(a)
$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix} \cdot \mathbf{x}.$$

(b)
$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \cdot \mathbf{x}.$$