PILOT

Problem Set 11

Differential Equations

Fall 2025

In the digressions to non-linear system, we are gradually seeing more dynamical systems and applications to the real life situations. As we explore cases with the non-linear system, please keep in mind that we are always trying to find a linear case to model as we zoom in. At the same moment, please take a moment to review what we have learned together.

- Concepts:
  - Nonlinear System
  - Locally Linear System
  - Jacobian Matrix
- Models:
  - Predator-Pray Model
  - Competing Species Model
  - Limit Cycles*

Tip of the Week

Study with the help of a Learning Den Tutor? Build a finals study schedule with a study consultant? CSS Saturday is on Nov. 15, Dec. 6, and Dec 13 at 12–4pm.

Course supported: Introductory Chemistry I, Organic Chemistry I, Physics I & II, PreCalculus, Calulus I, II, II, Gateway Computing: Java, Python.

1. (System with Unknown Coefficients). Let a non-linear system for $x = x(t)$ and $y = y(t)$ be:

$$\begin{cases} x' = \alpha x - y + y^2, \\ y' = x + \alpha y. \end{cases}$$

   (a) Show that $(0, 0)$ is a critical point, and show system is locally linear at $(0, 0)$ for all $\alpha \in \mathbb{R}$.

   (b) Classify the critical point $(0, 0)$ and sketch a few phase portraits of the linearized system.

2. (Critical Point). Find all the critical point in the following first order system:

$$\begin{cases} x' = 2x^3 - x^2 - 4x + 3 - y^2, \\ y' = 2x - y. \end{cases}$$

3. (Nonlinear at origin). Let the linear system be:

$$\begin{cases} x' = y, \\ y' = x + 2x^3. \end{cases}$$

   (a) Show that the origin is a saddle point.

   (b) Sketch a phase portrait for the linearized system. Note that where all the trajectories of the linear system tend to the origin.

4. (Modeling Politics). Suppose D and R are two parties on a non-existing country on the center of Mars. For the simplicity of this problem, they, *unfortunately*, have no elections. Therefore, we can model the amount of the supporter for each party (in millions), denoted x_D and x_R with the following relationship:

$$\begin{cases} \dfrac{dx_D}{dt} = x_D(1 - x_D - x_R), \\ \dfrac{dx_R}{dt} = x_R(3 - 2x_D - 4x_R). \end{cases}$$

Find all possible endings (say arbitrarily long after, that is $t \to \infty$) of the number of supporters (in millions) for the two parties.