PILOT

Problem Set 13: Solutions

Differential Equations

Fall 2025

Attention on Different Contents.

The questions labeled with **S** are the questions designated for Series Solutions for Second Order ODEs (Section 01-03) and the questions labeled with **L** are the questions designated for Laplace Transformation (Section 04-06).

We are almost at the end of the semester. As we are wrapping up the final materials, please be proficient on the one of the topics below on your own choice.

Series Solution for Second Order ODEs:

- Concepts:
  - Analytic at a Point
  - Pointwise Convergence
  - Absolute Convergence
  - Ordinary/Singular Point
  - Regular/Irregular Singularity
- Methodologies:
  - Recurrence Relationship
  - Euler's Equations

Laplace Transformation:

- Concepts:
  - Laplace Transformation
  - Step & Impulse Function
  - Convolution
- Methodologies:
  - Inverse Transformation
  - Shifting Theorems
  - IVP Solutions

The last topic of the semester would be on numerical approximation methods, *i.e.*, finding a rough approximation of complex models.

Euler's Method:

- Methodologies:
  - Numerical Methods
  - Under/Overestimate
  - Higher Improvements

Hope that you are enjoying this course up to right now, and enjoying the PILOT learning for the course.

Clubs & Orgs Bulletin

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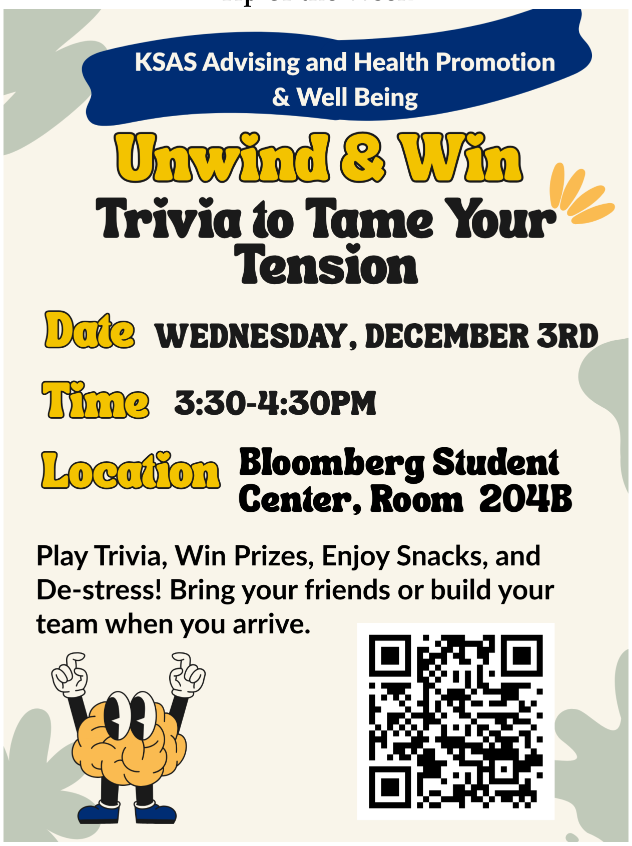
Studio North Are you interested in gaining hands-on film production experience and want to meet other film fans? Come to Studio North, JHU's student-run production club! Join our slack at https://tr.ee/XGlXuc7Mcv for updates on GBMs and workshops, and follow our Instagram `@studionorthmd`!

Motorsports Society Are you interested in motorsport, whether you're an F1 fan, becoming a mechanical engineer, or pursuing a career in sports journalism? Join Motorsoc! We foster a fun, inclusive environment for all kinds of motorsport fans. Follow our Instagram `@motor_socjhu` for more updates!

Tip of the Week



1. (Dispersion of Heat). For this problem, we consider the dispersion of heat for an object in an environment with fixed temperature. Here, let $\theta := \theta(t)$ be the temperature of the object and θ_0 denote the fixed temperature of the environment, we may model the temperature of the object by:

$$\frac{d\theta}{dt} = -\frac{1}{\kappa}(\theta - \theta_0),$$

where κ is a fixed positive constant, representing the rate of heat dispersion.

Suppose that we have a rigid body of 100°C (equivalently 212°F), and the room temperature is fixed as 20°C (equivalently 68°F, and this is also condition for STP, standard temperature and pressure). Also, we assume that $\kappa = 2$.

(a) Construct the differential equation for the above system.

(b) Use *Euler's method* with step size of 1 to approximate the temperature at $t = 3$.

(c) Identify if the approximation of temperature is an underestimate or an overestimate.

S2. (Logarithms and Recurrence Relations). We aim to solve the following differential equation:

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = 0.$$

using recurrence relationship.

(a) Write down the power series of $\log(x+1)$.

(b) Find the *recurrence relationship* for the differential equation.

(c) Find the fundamental set of solutions for the differential equation.
Hint: Make a conjecture from a pattern of the first few terms.

L2. (Step Functions). Given a piecewise defined function as follows:

$$f(t) = \begin{cases} 0, & \text{when } t < 0, \\ t^2, & \text{when } 0 \le t \le 2, \\ 2 + t, & \text{when } t > 2. \end{cases}$$

(a) Write $f(t)$ in terms of step functions.

(b) Find the Laplace transformation of $f(t)$.

(c) Find the general function $g(t)$ such that $g'(t) = f(t)$ almost everywhere (except the points that are not differentiable).

S3. (Euler's Equations). Let a differential equation of $y := y(x)$ defined as:

$$x^2y'' + xy' + cy = 0,$$

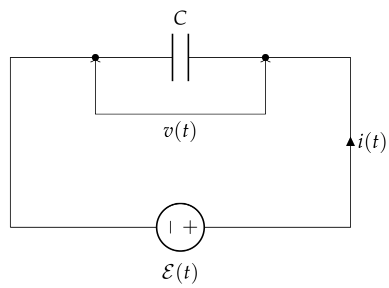
where $c \in \mathbb{R}$ is a fixed constant, we want to solve the differential equation using *Euler's equations*.

(a) Assume $c = -4$, solve the solution to the differential equation.

(b) Assume $c = 9$, solve the solution to the differential equation.

(c) Find the critical point to this system where the behavior of the solution changes.

L3. (Complex Impedance). Capacitors play an important role in many electronic circuits (as it can store the EMF from the source). Given a capacitor with constant capacitance C, on a simple circuit (as below). We can model the current flow (denoted $i(t)$) and the voltage around the capacitor (denoted $v(t)$) as functions of time.



Here, we can model the voltage around the capacitor using the following differential equation:

$$\begin{cases} i(t) = C\dfrac{dv}{dt}, \\ v(0) = 0. \end{cases}$$

The **complex impedance** $Z(s)$ is the frequency related version of the resistance, which is defined as:

$$Z(s) = \frac{\mathcal{L}\{v\}(s)}{\mathcal{L}\{i\}(s)}.$$

(a) Compute the complex impedance $Z(s)$.

(b) Suppose that we observe the current flow $i(t)$ and notice the following:

$$i(t) = \begin{cases} 0, & \text{when } t < 0 \text{ and } t > 5, \\ 1, & \text{when } 0 \leq t \leq 5. \end{cases}$$

Express $i(t)$ as step functions and find the voltage using Laplace transformation.

S4. (Singularities, Zeros, and Poles). For any function $f : \mathbb{C} \to \mathbb{C}$, and $z_0 \in \mathbb{C}$, we have the following:

- It has **a zero of order** $\boldsymbol{m}$ at z_0 if $f(z_0) = 0$, and m is the smallest positive integer such that $f(z) = (z - z_0)^m g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.
- It has **a pole of order** $\boldsymbol{n}$ at z_0 if $f(z_0)$ is not defined, and n is the smallest integer such that $g(z) = (z - z_0)^n f(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.
- If a zero/pole has order 1, it is **simple**.

As a side note, such definition applies for any real valued functions, *i.e.*, $f : \mathbb{R} \to \mathbb{R}$.
Here, we define a differential equation for $y := y(x)$ as:

$$\sin(x)y'' + \sin(x)\big(\cos(x) - e^x + x\big)y' + \big(\csc(x)\big)y = 0$$

(a) Write the differential equation in the form of:

$$y'' + p(x)y' + q(x)y = 0.$$

(b) Identify all zeros and poles of $p(x)$ and $q(x)$ as real functions, *i.e.*, $p, q : \mathbb{R} \to \mathbb{R}$. Find the order of the zeros and poles.

(c) Identify all the points $x_0 \in \mathbb{R}$ such that the differential equation has a *regular singular point*.

L4. (Second Order Non-homogeneous Step Function). Solve for the general solution to the following second order non-homogeneous differential equation:

$$y'' + 2y' + y = u_5(x) - u_2(x).$$

Hint: Use Laplace transformation on a specific initial value problem.