PILOT

Problem Set 5

Differential Equations

Fall 2025

Welcome to the realm of second order differential equations. As we just got into this part, we will explore the basic cases as well as some foundations of the second order differential equations. For this problem set, we will briefly investigate two core ideas, the linear independence and Euler's theorem.

Tip of the Week

Did you know that Hopkins has a planetary observatory right on Homewood campus? Every Friday night the Maryland Space Grant Observatory located on the roof of Bloomberg hosts a free open house for the public after sunset, weather permitting. Check here to see updates on hours and visibility status https://md.spacegrant.org/observatory-open-house/.

1. (Second Order Differential Equations). Find the general solution to the following second order differential equations on $y := y(t)$. Then verify using Wronskian that the solutions are linearly independent.

   (a) $$y'' - 3y' + 2y = 0.$$

   (b) $$y'' + 12y' - 3y = 0.$$

2. (Second Order IVP). Let an initial value problem for $y = y(t)$ be defined as follows:

$$\begin{cases} 4y'' - y = 0, \\ y(0) = 2,\ y'(0) = \beta, \end{cases}$$

where β is a real constant.

(a) Find the specific solution to the initial value problem. Express your solution with constant β.

(b) Find the value of β such that the solution *converges* to 0 as t tends to infinity.

3. (Euler's Theorem). In our study of differential equations, our main focus is on *real-valued functions*. But we are about to see complex numbers in our story. **Euler's theorem** states that for any $z \in \mathbb{C}$, we have:

$$\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z).$$

   (a) To review on complex numbers, compute/simplify the following expressions:

$$\text{(i)}(2+5\mathrm{i}) \times (1+2\mathrm{i}), \qquad \text{(ii)}\frac{2-3\mathrm{i}}{1+\mathrm{i}}, \qquad \text{(iii)}\overline{2+5\mathrm{i}}, \qquad \text{(iv)}(20+25\mathrm{i}) \times (\overline{20+25\mathrm{i}}).$$

   (b) Write the following complex exponentials in terms of a sum of the real and imaginary parts:

$$\text{(i)}\exp(\mathrm{i}), \qquad \text{(ii)}\exp\left(\frac{\pi\mathrm{i}}{3}\right), \qquad \text{(iii)}\exp(2+2\mathrm{i}).$$

4. (Second Order IVP with Dirichlet Condition). Given following initial value problem.

$$\begin{cases} y'' = -y, \\ y(0) = 0, \\ y(b) = 0, \end{cases}$$

where $b \in \mathbb{R}$ and $b \neq 0$.

(a) Find the general solution to $y'' = -y$.

(b) Suppose $b = \frac{\pi}{4}$, find the solution.

(c) Give a value of b such that there is not a unique solution.