PILOT Quiz 2 Review

Differential Equations

Johns Hopkins University

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As you prepare for quiz 2, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/FA25/`
  - Find the review problem sets for quiz 2.
  - Consult the archives page for PILOT sets from the semester.
- Review the *homework/quiz sets* provided by the instructor.
- Join the PILOT Quiz 2 Review Session. (You are here.)

Plan for today:

1. Go over all contents that we have covered for this semester so far.
2. In the end, we will open the poll to you. Please indicate which problems from the Review Set that you want us to go over.

Part 1:
Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

First Order ODEs

- Exactness Problem
  - Integrating Factor for Non-Exact Case
- Bifurcation
  - Bifurcation Diagram

The condition for a function in form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ to be exact is:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

For solving Exact ODEs, either finding $\int M(x,y)dx + h(y)$ or $\int N(x,y)dy + h(x)$ and taking partials again to fit gives the solution $\Psi(x,y) = C$.

Integrating Factor for Non-Exact Case

$$\mu(t) = \exp\left(\int \frac{M_y - N_x}{N} dx\right) \text{ or } \mu(t) = \exp\left(\int \frac{N_x - M_y}{M} dy\right).$$

When a differential equation contains some unknown, fixed parameter C, its equilibriums would exhibit different behavior, the bifurcation value is the critical value such that the equilibriums have different stability.

Bifurcation Diagram

A bifurcation diagram is the vertical concatenation of phase portraits (C-y plot), in which the equilibriums will be marked for respective values of C.

Second Order ODEs

- Linear Homogeneous Cases
  - Complex Characteristic Roots
  - Repeated Characteristic Roots
- Linear Independence
  - Definition of Linearly Independence
  - Superposition Principle
- Reduction of Order
  - Product Rule and Chain Rule

Consider the linear homogeneous ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is $r^2 + pr + q = 0$, with real, distinct solutions r_1 and r_2, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t} \cos(\beta t) + c_2 e^{\lambda t} \sin(\beta t).$$

Repeated Characteristic Roots

If the solutions are repeated, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

To form a fundamental set of solutions, the solutions need to be linearly independent, in which the Wronskian (W) must be non-zero, meaning that:

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \cdots, f_n, \cdots\}$ is linearly independent when for $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots \in \mathbb{F}$ (typically $\mathbb{C}$):

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n + \cdots = 0 \iff \lambda_1 = \lambda_2 = \cdots = \lambda_n = \cdots = 0.$$

Superposition Principle

If $y_1(t)$ and $y_2(t)$ are solutions to $l[y] = 0$, then the solution $c_1 y_1(t) + c_2 y_2(t)$ are also solutions for all constants $c_1, c_2 \in \mathbb{R}$.

For non-linear second order homogeneous ODEs, when one solution $y_1(t)$ is given, the other solution is in form:

$$y_2(t) = u(t) \cdot y_1(t).$$

Product Rule and Chain Rule

- **Product Rule**: $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x)g(x) + f(x)\frac{dg}{dx}(x)$.
- **Chain Rule**: $\frac{d}{dx}\left[f(g(x))\right] = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x)$.

Procedure of Reduction of Order

As long as $y_1(t)$ is a solution, you will be able to reduce the differential equation with respect to y_2 into a differential equation involving only $u''(t)$ and $u'(t)$ terms to solve for $\omega(t) = u'(t)$.

Part 2:
Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through selected problems from the practice problem set.
- We are also open to conceptual questions with the course.

Good luck on your second exam.