PILOT Quiz 3 Review

Differential Equations

Johns Hopkins University

Fall 2025

As you prepare for quiz 3, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/FA25/`
  - Find the review problem sets for quiz 3.
  - Consult the archives page for PILOT sets from the semester.
- Review the *homework/quiz sets* provided by the instructor.

Part 1:
Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

1 Second Order ODEs (Continued)

2 Higher Order ODEs

3 System of First Order Linear ODEs

Second Order ODEs (Continued)

- Non-homogeneous Cases
  - Variation of Parameters
  - Undetermined Coefficients

Let the differential equation be:

$$Ay''(t) + By'(t) + Cy(t) = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$ and $y_2(t)$ be the two homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

Variation of Parameters

The particular solution of the differential equation can be written as the integrals of respective parts.

$$y_p = y_1(t) \int \frac{-y_2(t) \cdot g(t)}{W} dt + y_2(t) \int \frac{y_1(t) \cdot g(t)}{W} dt.$$

Another approach is less calculation intensive, but requires the function $g(t)$ to be constrained in certain forms.

Undetermined Coefficients

A guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \cos(ax)$
Exp.:	e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

Higher Order ODEs

- Existence and Uniqueness Theorem
- Homogeneous Cases
  - Complex Characteristic Roots
  - Repeated Characteristic Roots
- Linear Independence
  - Definition of Linearly Independence
- Abel's Formula
- Non-Homogeneous Cases
  - Variation of Parameters
  - Undetermined Coefficients

For higher order IVP in form:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots\ , y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

If $P_0(t), P_1(t), \cdots, P_{n-1}(t)$, and $g(t)$ are continuous on an interval I containing t_0. Then there exists a unique solution for $y(t)$ on I.

Only Contrapositive is Guaranteed to be True

Again, for this theorem, you can conclude that if *there does not exist a solution or the solution is not unique*, then *the conditions must not be satisfied*. You **cannot** conclude that if *the conditions are not satisfied*, then *there is no unique solution*.

The higher order homogeneous ODEs are in form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

By computing the characteristic equation $r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$, with solutions $r_1, r_2, \cdots, r_n$, the general solution is $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$.

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t}\cos(\beta t) + c_2 e^{\lambda t}\sin(\beta t) + \text{ rest of the solutions.}$$

Repeated Characteristic Roots

If the solutions are repeated with multiplicity m, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} + \cdots c_m t^{m-1} e^{rt} + \text{ rest of the solutions.}$$

To obtain the fundamental set of solutions, the Wronskian (W) must be non-zero, where Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \cdots, f_n, \cdots\}$ is linearly independent when for $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots \in \mathbb{F}$ (typically $\mathbb{C}$):

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n + \cdots = 0 \iff \lambda_1 = \lambda_2 = \cdots = \lambda_n = \cdots = 0.$$

For higher order ODEs in the form of:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots, y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

Its Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = Ce^{\int -P_{n-1}(t)dt},$$

where C is independent of t but depend on $y_1, y_2, \cdots, y_n$.

Let the differential equation be:

$$L[y^{(n)}(t), y^{(n-1)}(t), \cdots, y(t)] = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$, $y_2(t)$, $\cdots$, $y_n(t)$ be all homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

Variation of Parameters

The particular solution is:

$$y_p = y_1(t)\int \frac{W_1 g}{W}dt + y_2(t)\int \frac{W_2 g}{W}dt + \cdots + y_n(t)\int \frac{W_n g}{W}dt,$$

where W_i is defined to be the Wronskian with the i-th column alternated into $\begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^\intercal$.

Undetermined Coefficients

Same as in degree 2, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \cos(ax)$
Exp.:	e^{at}	Ce^{at}

Again, the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

System of First Order Linear ODEs

- Solving for Eigenvalues and Eigenvectors
- Linear Independence
  - Abel's Formula

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - rI) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - rI) \cdot \boldsymbol{\xi} = \mathbf{0}.$$

Suppose that the eigenvalues are distinct and the eigenvectors are linearly independent, the solution to the ODE is:

$$\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t} + \cdots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}.$$

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det(\Psi(t)).$$

The system is linearly independent if the Wronskian is non-zero.

Abel's Formula

For the linear system in form:

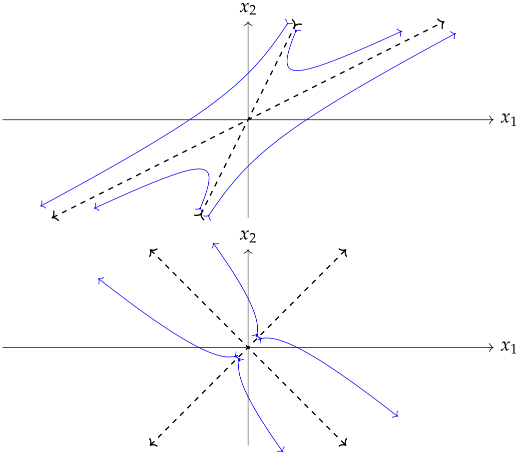
$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A, which is the sum of the diagonals, that is:

$$W = Ce^{\int \text{trace}\, A dt} = Ce^{\int (A_{1,1}+A_{2,2}+\cdots+A_{n,n})dt}.$$

In particular, we can sketch the linear system of $\mathbb{R}^2$ in terms of phase portraits given the eigenvalues and eigenvectors.

- For a node graph, we have it as (directions might vary):



Good luck on your third exam.