PILOT Quiz 4 Review

Differential Equations

Johns Hopkins University

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As you prepare for quiz 4, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/FA25/`
  - Find the review problem sets for quiz 4.
  - Consult the archives page for PILOT sets from the semester.
- Review the *homework/quiz sets* provided by the instructor.

Part 1:
Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

System of First Order Linear ODEs (Continued)

- Repeated Eigenvalues
  - Algebraic Multiplicity and Geometric Multiplicity
- Phase Portraits
  - Node Graph
  - Spiral/Center Graph
  - Repeated Eigenvalue Graph

For repeated eigenvalue r with only one (linearly independent) eigenvector, if a given a solution is $\mathbf{x}^{(1)} = \boldsymbol{\xi} e^{rt}$, the other solution would be:

$$\mathbf{x}^{(2)} = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt},$$

where $(A - Ir) \cdot \boldsymbol{\eta} = \boldsymbol{\xi}$, and $\mathbf{x}^{(2)}$ is called the *generalized eigenvector*.

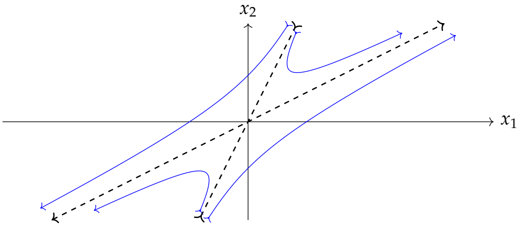
Algebraic Multiplicity and Geometric Multiplicity

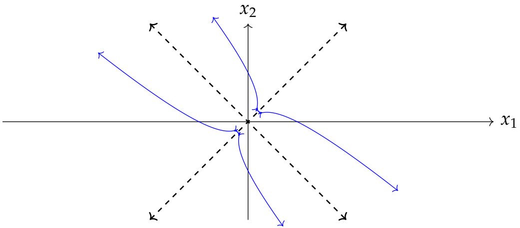
The algebraic multiplicity refers to the multiplicity of root in the characteristic polynomial, and the geometric multiplicity refers to the dimension of the eigenspace associated with the eigenvalue.

- The algebraic multiplicity will be no less than the geometric multiplicity for each eigenvalue.
- We need the generalized eigenvector when the algebraic multiplicity is larger than the geometric multiplicity.

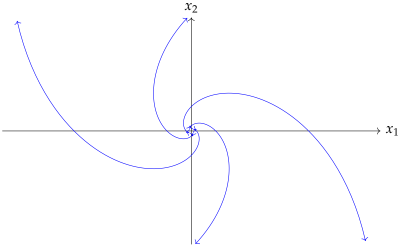
In particular, we can sketch the linear system of $\mathbb{R}^2$ in terms of phase portraits given the eigenvalues and eigenvectors.

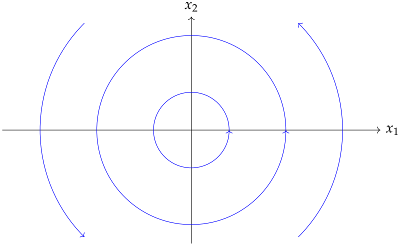
- For a node graph, we have it as (directions might vary):



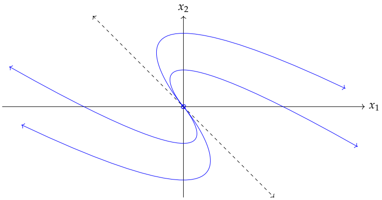


- For a spiral/center graph, we have it as (directions might vary):

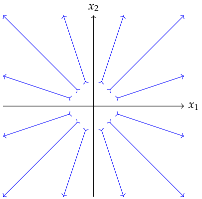




- For repeated eigenvalues with less geometric multiplicity, the solution is (directions might vary):



- If the geometric multiplicity is the same, the graph is simply a radial shape (directions might vary):



Non-linear Systems

- Linear Approximation
  - Autonomous Systems
- Stability
- Limit Cycles

For non-linear system $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}'$, if $F, G \in C^2$ and the system is locally linear, the approximation at critical point (x_0, y_0) is:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \mathbf{J}(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where Jacobian is:

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}.$$

Autonomous Systems

When $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} F(y) \\ G(x) \end{pmatrix}$, it can be solved implicitly for:

$$\frac{dy}{dx} = \frac{G(x)}{F(y)}.$$

For linearized system with eigenvalues r_1, r_2, the stability can be concluded as follows:

Eigenvalues	**Linear System**		**Nonlinear System**	
	Type	**Stability**	**Type**	**Stability**
Eigenvalues are λ_1 and λ_2				
$0 < \lambda_1 < \lambda_2$	Node	Unstable	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable	Node or Spiral Point	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable	Node or Spiral Points	Asymptotically Stable

Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$				
$\alpha > 0$	Spiral Point	Unstable	Spiral Point	Unstable
$\alpha = 0$	Center	Stable	Center or Spiral Point	Indeterminate
$\alpha < 0$	Spiral Point	Asymptotically Stable	Spiral Point	Asymptotically Stable

A closed trajectory or periodic solution repeats back to itself with period τ:

$$\begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Closed trajectories with either side converging to/diverging from the solution is a limit cycle.

Conversion to Polar Coordinates

A Cartesian coordinate can be converted by:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ rr' = xx' + yy', \\ r^2\theta' = xy' - yx'. \end{cases}$$

For a linear system $x = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$ with $F, G \in C^1$:

1. A closed trajectory of the system must enclose at least 1 critical point.
2. If it only encloses 1 critical point, then that critical point cannot be saddle point.
3. If there are no critical points, there are no closed trajectories.
4. If the unique critical point is saddle, there are no trajectories.
5. For a simple connected domain D in the xy-plane with no holes. If $F_x + G_y$ had the same sign throughout D, then there is no closed trajectories in D.

Laplace Transformation

- Laplace Transformation
  - Properties of Laplace Transformation
- Elementary Laplace Transformations

The Laplace Transformation of a function f is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Properties of Laplace Transformation

1. Laplace Transformation is a linear operator: $\mathcal{L}\{f + \lambda g\} = \mathcal{L}\{f\} + \lambda \mathcal{L}\{g\}$.
2. Laplace Transformation for derivatives:

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0), \\
\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0), \\
&\vdots \\
\mathcal{L}\{f^{(n)}(t)\} &= s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0).
\end{aligned}$$

3. First Shifting Theorem: $\mathcal{L}\{e^{ct} f(t)\} = F(s - c)$.

Here are the Laplace Transformation of some elementary functions, which can also be calculated by definition:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$t^n, n \in \mathbb{Z}_{>0}$	$\frac{n!}{s^{n+1}}, s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}, s > 0$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$

Laplace Transformations can be used for solving IVP, with the derivative rules and inverse operation.

Series Solutions to Second-Order Linear Equations

- Power Series

A power series is an infinite series in the form:

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots,$$

where a_n is the coefficient for term n and c is the center of the approximation.

A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *converge* at a point x if:

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-x_0)^n \text{ exists for that } x.$$

A power series *converges pointwise* on X if it converges on every $x \in X$.

A power series converges absolutely at a point x if the power series:

$$\sum_{n=0}^{\infty} |a_n(x - x_0)^n| = \sum_{n=0}^{\infty} |a_n||x - x_0|^n \text{ converges.}$$

Note that absolute converges implies convergence, but the converse is not true.

Here are some properties of series:

1. (Ratio test). If $a_n \neq 0$, and if for a fixed value of x, and:
$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{x_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L,$$
then the power series converges absolutely at x if $|x-x_0|L < 1$ and diverges if $|x-x_0|L > 1$.
2. (Monotonic property). If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at $x = x_1$, then it converges absolutely for $|x-x_0| < |x_1-x_0|$. If it diverges at $x = x_1$, then it diverges for $|x-x_0| > |x_1-x_0|$.
3. (Radius of convergence). Let $\rho > 0$ be such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$, then ρ is the *radius of convergence* and $(x_0-\rho, x_0+\rho)$ is the *interval of convergence*.

Also, we note that power series can be added or subtracted term-wise. They can also be multiplied and divided by having divisions of terms.

Recall that by Taylor theorem, suppose $f \in C^\infty$, then we can form the Taylor polynomial as a power series, with coefficient:

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

In particular, if f has a Taylor polynomial at x_0 with a positive radius of convergence, we say the series is *analytic* at x_0.

Good luck on your fourth exam.