

PILOT Final Review

Differential Equations

Johns Hopkins University

Spring 2025

As you prepare for the final, please consider the following resources:

- PILOT webpage for ODEs:
<https://jhu-ode-pilot.github.io/SP25/>
 - Since the final is cumulative, all problem sets will be relevant.
 - Find the review problem set for final exam, as well as review problem sets for midterm 1 and midterm 2.
- Review the *homework sets* provided by the instructor.
- Join the PILOT Final Review Session. (You are here.)

Plan for today:

- 1 Go over all contents that we have covered for this semester so far.
- 2 In the end, we will open the poll to you. Please indicate which problems from the PSets or Review Set that you want us to go over.

Contents:

1 System of First Order Linear ODEs

- Solving for Eigenvalues and Eigenvectors
- Linear Independence
- Abel's Formula
- Repeated Eigenvalues
- Phase Portraits

2 Non-linear Systems

- Linear Approximation
- Autonomous Systems
- Stability
- Limit Cycles
- Conversion to Polar Coordinate

3 Review Problems

Part 1:

Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - Ir) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - Ir) \cdot \boldsymbol{\xi} = \mathbf{0}.$$

The solution to the ODE is:

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + c_2 \boldsymbol{\xi}^{(2)} e^{r_2 t} + \cdots + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}.$$

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det(\Psi(t)).$$

The system is linearly independent if the Wronskian is non-zero.

For the linear system in form:

$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A , which is the sum of the diagonals, that is:

$$W = Ce^{\int \text{trace } A dt} = Ce^{\int (A_{1,1} + A_{2,2} + \cdots + A_{n,n}) dt}.$$

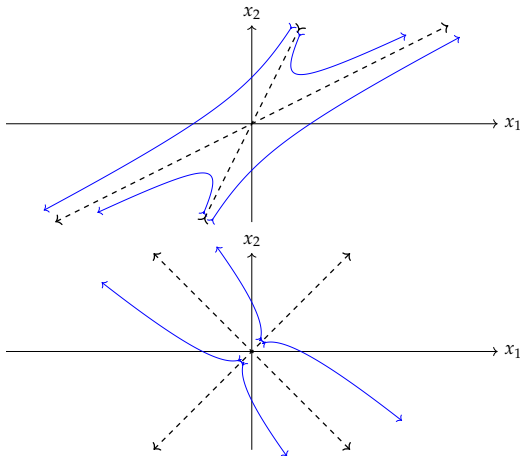
For repeated eigenvalue r with only one eigenvector, if a given solution is $\mathbf{x}^{(1)} = \boldsymbol{\zeta}e^{rt}$, the other solution would be:

$$\mathbf{x}^{(2)} = \boldsymbol{\zeta}te^{rt} + \boldsymbol{\eta}e^{rt},$$

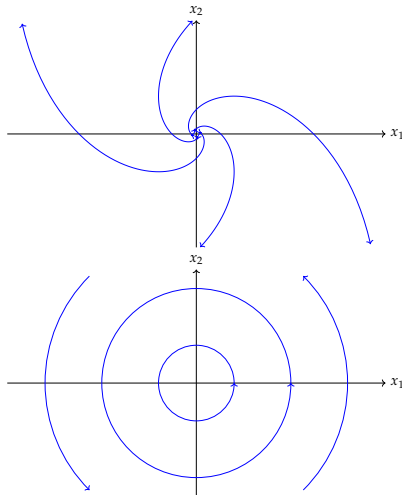
where $(A - Ir) \cdot \boldsymbol{\eta} = \boldsymbol{\zeta}$.

In particular, we can sketch the linear system of \mathbb{R}^2 in terms of phase portraits given the eigenvalues and eigenvectors.

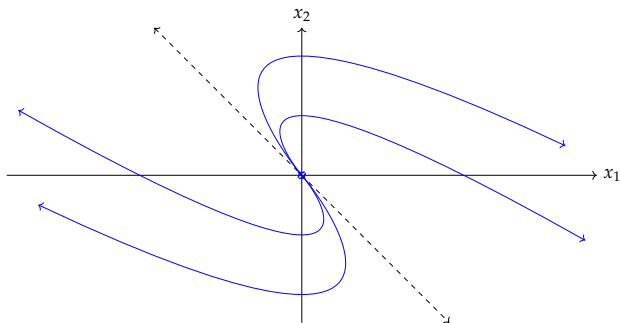
- For a node graph, we have it as (directions might vary):



- For a spiral/center graph, we have it as (directions might vary):



- For repeated eigenvalues, the solution depends is (directions might vary):



For non-linear system $\mathbf{x}' = \begin{pmatrix} F \\ G \end{pmatrix} \mathbf{x}$, if $F, G \in C^2$, i.e. locally linear, the approximation at critical point (x_0, y_0) is:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \mathbf{J}(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where Jacobian is:

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}.$$

When $\mathbf{x} = \begin{pmatrix} F(y) \\ G(x) \end{pmatrix}$, it can be solved implicitly for:

$$\frac{dy}{dx} = \frac{G(x)}{F(y)}.$$

For linearized system with eigenvalues r_1, r_2 :

- 1 If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$: $r_1 < r_2 < 0$ indicates an asymptotically stable node, $r_1 < 0 < r_2$ indicates a mostly unstable saddle, and $0 < r_1 < r_2$ indicates an unstable node. Note that these will not change for the non-linear case.
- 2 If $r_1 = r_2$: $r_1 = r_2 < 0$ indicates a asymptotically stable node and $r_1 = r_2 > 0$ indicates an unstable node. The stability preserves but the shape either node or spiral.
- 3 If $r_1, r_2 \in \mathbb{C}$ and $\text{Re}(r_1) = \text{Re}(r_2) \neq 0$: $\text{Re}(r_1) = \text{Re}(r_2) > 0$ indicates an unstable spiral and $\text{Re}(r_1) = \text{Re}(r_2) < 0$ indicates an asymptotically stable spiral. Note that these will not change for the non-linear case.
- 4 If $r_1, r_2 \in \mathbb{C}$ and $\text{Re}(r_1) = \text{Re}(r_2) = 0$: That indicates a stable center. In the non-linear case, the shape is either spiral or center, but the stability is in-determinant.

The stability can be concluded as follows:

Eigenvalues	Linear System		Nonlinear System	
	Type	Stability	Type	Stability
Eigenvalues are λ_1 and λ_2				
$0 < \lambda_1 < \lambda_2$	Node	Unstable	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable	Node or Spiral Point	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable	Node or Spiral Points	Asymptotically Stable
Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$				
$\alpha > 0$	Spiral Point	Unstable	Spiral Point	Unstable
$\alpha = 0$	Center	Stable	Center or Spiral Point	Indeterminate
$\alpha < 0$	Spiral Point	Asymptotically Stable	Spiral Point	Asymptotically Stable

A closed trajectory or periodic solution repeats back to itself with period τ :

$$\begin{pmatrix} x(t + \tau) \\ y(t + \tau) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Closed trajectories with either side converging to/diverging from the solution is a limit cycle.

A Cartesian coordinate can be converted by:

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \\ rr' = xx' + yy', \\ r^2\theta' = xy' - yx'. \end{cases}$$

For a linear system $x = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}$ with $F, G \in C^1$:

- 1 A closed trajectory of the system must enclose at least 1 critical point.
- 2 If it only encloses 1 critical point, then that critical point cannot be saddle point.
- 3 If there are no critical points, there are no closed trajectories.
- 4 If the unique critical point is saddle, there are no trajectories.
- 5 For a simple connected domain D in the xy -plane with no holes. If $F_x + G_y$ had the same sign throughout D , then there is no closed trajectories in D .

Part 2: Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through the practice problem set sequentially.
- We are also open to conceptual questions with the course.

- 1 Let systems of differential equations be defined as follows, find the general solutions to the equations:

$$(a) \quad \mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = (x_1, x_2).$$

$$(b) \quad \mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = (x_1, x_2).$$

$$(c) \quad \mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = (x_1, x_2, x_3).$$

- 2 Solve the following initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

- 3 For the following non-linear systems, find all equilibrium(s) and classify their stability locally.

$$(a) \quad \begin{cases} \frac{dx}{dt} = x - y^2, \\ \frac{dy}{dt} = x + x^2 - 2y. \end{cases}$$

$$(b) \quad \begin{cases} \frac{dx}{dt} = 2x + 3y^2, \\ \frac{dy}{dt} = x + 4y^2. \end{cases}$$

- 4 Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:

$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$

Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.

- 1 Find the set of all equilibrium(s) for \mathbf{x} .
- 2 Find the set in which the equilibrium(s) is locally linear.
Now, $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is not necessarily well-behaved.
- 3 Construct a function F such that \mathbf{x} has a equilibrium that is not locally linear.

Hint: Consider the condition in which a non-linear system is locally linear.

- 5 Let the following systems of (x, y) be functions of variable t :

$$(a) \quad \begin{cases} x' = (1 + x) \sin y, \\ y' = 1 - x - \cos y. \end{cases}$$

$$(b) \quad \begin{cases} x' = x - y, \\ y' = x - 2y + x^2. \end{cases}$$

Identify the corresponding linear system, then evaluate the stability for the equilibrium at $(0, 0)$ by showing it is locally linear.

- 6 Determine the periodic solution, if there are any, of the following system:

$$\begin{cases} x' = y + \frac{x}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2), \\ y' = -x + \frac{y}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2). \end{cases}$$

Good luck on your final exam.

**Thank you for being with PILOT for Differential Equations
this semester. We wish the best of all for you future
academic and career pursuits!**