PILOT

Problem Set 10

Differential Equations

Spring 2025

We hope that your preparation for the second midterm is smooth and successful. The problem set of this week would cover some hardest topics since the last midterm. In linear systems, you should be familiar with the real eigenvalue cases, and we shall attempt work on the case of complex eigenvalues.

Tip of the Week

The National Cherry Blossom Festival takes place in Washington DC from March 20th to April 13th this year! MARC train tickets from Baltimore Penn Station to DC are available for $9 and can be purchased on the CharmPass app. Find out more here: https://nationalcherryblossomfestival.org/.

1. (Non-homogeneous Solutions). Find the general solution to the following differential equations:
   (a) $$y''' - 4y' = e^{-2t}.$$
   (b) $$y'' + 36y = e^{t}\sin(6t).$$

2. (Euler's Equations). Find the full set of solutions for the following second order ODEs, given one solution:

$$x^2y'' + xy' - 4y = 0, \qquad y_1(x) = x^2.$$

3. (Solving Linear Systems). Let $\mathbf{x} \in \mathbb{R}^2$, find the general solution of $\mathbf{x}$ for:

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \cdot \mathbf{x}.$$

4. (Linear Systems with Complex Eigenvalues). We will work step-by-step for the construction of a complex eigenvalue problem. Consider $\mathbf{x} \in \mathbb{R}^2$, and the differential equation be:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{x}.$$

   (a) Find the eigenvalues and eigenvectors of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   (b) Use Euler's formula to expand a complex solution of one eigenvalue and its corresponding eigenvector to find a solution to the differential equation.

   (c) Verify that the real and imaginary part of the solution are both solutions to the differential equation, and verify that they are linearly independent.