Problem Set 10: Solutions

Differential Equations

Spring 2025

1. (Non-homogeneous Solutions). Find the general solution to the following differential equations:

   (a) $$y''' - 4y' = e^{-2t}.$$

   (b) $$y'' + 36y = e^t \sin(6t).$$

Solution:

(a) First, we find the homogeneous case, that is $y''' - 4y' = 0$, whose characteristic equation is $r^3 - 4r = 0$, so the roots are $r = 0, 2, -2$, hence the homogeneous solution is:

$$y(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t}.$$

Given that the non-homogeneous part already exists in the equation, then our guess should be $y_p(t) = Ate^{-2t}$, which the derivatives as:

$$y_p'(t) = Ae^{-2t} - 2Ate^{-2t},$$
$$y_p''(t) = -4Ae^{-2t} + 4Ate^{-2t},$$
$$y_p'''(t) = 12Ae^{-2t} - 8Ate^{-2t}.$$

Note that when we plug into our equation, we have:

$$(12Ae^{-2t} - 8Ate^{-2t}) - 4(Ae^{-2t} - 2Ate^{-2t}) = e^{-2t}.$$

Note that the te^{-2t} term vanishes (why?), we now have:

$$8Ae^{-2t} = e^{-2t},$$

so we have that $A = 1/8$, so our general solution is:

$$y(t) = \boxed{C_1 + C_2 e^{2t} + C_3 e^{-2t} + \frac{1}{8} te^{-2t}}.$$

(b) Again, we find the homogeneous case, which is $y'' + 36y = 0$, whose characteristic equation is $r^2 + 36 = 0$, so the roots are $\pm 6i$, and the homogeneous solution is:

$$y(t) = C_1 \sin(6t) + C_2 \cos(6t).$$

Now, we need to form our guess as $y_p(t) = Ae^t \sin(6t) + Be^t \cos(6t)$, we take the derivatives as:

$$y_p'(t) = Ae^t \sin(6t) + 6Ae^t \cos(6t) + Be^t \cos(6t) - 6Be^t \sin(6t),$$
$$y_p''(t) = Ae^t \sin(6t) + 12Ae^t \cos(6t) - 36Ae^t \sin(6t) + Be^t \cos(6t) - 12Be^t \sin(6t) - 36Be^t \cos(6t)$$
$$= (-35A - 12B)e^t \sin(6t) + (12A - 35B)e^t \cos(6t).$$

When plugged back into the differential equation, we have:

$$(-35A - 12B + 36A)e^t \sin(6t) + (12A - 35B + 36B)e^t \cos(6t) = e^t \sin(6t).$$

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Then, we have a system of linear equations as:

$$\begin{cases} A - 12B = 1, \\ 12A + B = 0. \end{cases}$$

This solves into $A = \frac{1}{145}$ and $B = -\frac{12}{145}$, so the general solution is:

$$y(t) = \boxed{C_1 \sin(6t) + C_2 \cos(6t) + \frac{1}{145} e^t \sin(6t) - \frac{12}{145} e^t \cos(6t)}.$$

At this moment, we highly encourage readers to consider why the particular guess did **not** have an additional order, and why both sine and cosine are included when the derivative operators are of even orders.

2. (Euler's Equations). Find the full set of solutions for the following second order ODEs, given one solution:

$$x^2y'' + xy' - 4y = 0, \qquad y_1(x) = x^2.$$

Solution:
Here, we guess our second solution as $y_2(x) = x^2u(x)$, and we take the derivatives using product rule:

$$\begin{aligned} y_2'(x) &= 2xu(x) + x^2u'(x), \\ y_2''(x) &= 2u(x) + 4xu'(x) + x^2u''(x). \end{aligned}$$

Then, we plug them into the differential equation correspondingly:

$$\begin{aligned} &x^2\big(2u(x) + 4xu'(x) + x^2u''(x)\big) + x\big(2xu(x) + x^2u'(x)\big) - 4\big(x^2u(x)\big) \\ &= 5x^3u'(x) + x^4u''(x) = xu''(x) + 5u'(x) = 0. \end{aligned}$$

Now, by letting $\omega(x) := u'(x)$, we have:

$$x\omega'(x) + 5\omega(x) = 0,$$

and you should note that this is separable, so:

$$\frac{d\omega}{\omega} = -5\frac{dx}{x},$$

and integrating both sides gives that:

$$\log|\omega| = -5\log|x| + C,$$

so we have $\omega = \tilde{C}x^{-5}$, and we integrate again to obtain that:

$$u = \int \tilde{C}x^{-5}dx = \tilde{\tilde{C}}x^{-4} + D,$$

and by simple multiplication, we obtain that:

$$y_2(x) = Ax^{-2} + Bx^2,$$

and the fundamental set of solutions are:

$$\boxed{\left\{\frac{1}{x^2}, x^2\right\}}.$$

This type of function is called an Euler's equations, and there is a special way of obtaining the set of solutions using a formula rather than using reduction of order.

3. (Solving Linear Systems). Let $\mathbf{x} \in \mathbb{R}^2$, find the general solution of $\mathbf{x}$ for:

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \cdot \mathbf{x}.$$

Solution:

Here, we find the characteristic equation as:

$$0 = \det \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - (-1)\cdot 3 = 8 - 6\lambda + \lambda^2 = (\lambda-2)(\lambda-4).$$

Hence, the eigenvalues are 2 and 4, and the eigenvalues, respectively, are:

(a) For $\lambda_1 = \boxed{2}$, we have $A - 2\,\mathrm{Id}$ as $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$, we want find $\boldsymbol{\xi}^{(1)}$ such that $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} .\boldsymbol{\xi}^{(1)} = \mathbf{0}$, that is $3\xi_1^{(1)} - \xi_2^{(1)} = 0$, so we have $\xi_2^{(1)} = 3\xi_1^{(1)}$, so we have $\boldsymbol{\xi}^{(1)} = \boxed{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$.

(b) For $\lambda_2 = \boxed{4}$, we have $A - 4\,\mathrm{Id}$ as $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$, we want find $\boldsymbol{\xi}^{(2)}$ such that $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} .\boldsymbol{\xi}^{(2)} = \mathbf{0}$, that is $\xi_1^{(2)} - \xi_2^{(2)} = 0$, so we have $\xi_2^{(2)} = \xi_1^{(2)}$, so we have $\boldsymbol{\xi}^{(2)} = \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

Hence, the solution to the linear system is:

$$\mathbf{x} = \boxed{C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}.$$

4. (Linear Systems with Complex Eigenvalues). We will work step-by-step for the construction of a complex eigenvalue problem. Consider $\mathbf{x} \in \mathbb{R}^2$, and the differential equation be:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{x}.$$

   (a) Find the eigenvalues and eigenvectors of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   (b) Use Euler's formula to expand a complex solution of one eigenvalue and its corresponding eigenvector to find a solution to the differential equation.

   (c) Verify that the real and imaginary part of the solution are both solutions to the differential equation, and verify that they are linearly independent.

Solution:

(a) Here, we use our conventional method that:

$$\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0,$$

hence, the eigenvalues are $\pm\mathrm{i}$.
Then, we look for the eigenvectors, as usual.

   i. For the eigenvalue of $\boxed{\mathrm{i}}$, we have:

$$\begin{pmatrix} -\mathrm{i} & 1 \\ -1 & -\mathrm{i} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0},$$

   so we can derive out that $x_2 = \mathrm{i}x_1$, so the eigenvector is $x_1 \boxed{(1, \mathrm{i})}$.

   ii. For the eigenvalue of $\boxed{-\mathrm{i}}$, we have:

$$\begin{pmatrix} \mathrm{i} & 1 \\ -1 & \mathrm{i} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0},$$

   so we can derive out that $x_2 = -\mathrm{i}x_1$, so the eigenvector is $x_1 \boxed{(1, -\mathrm{i})}$.

(b) For this problem, we choose $\lambda = \mathrm{i}$ and $\boldsymbol{\xi} = (1, \mathrm{i})$, we have the solution as:

$$\mathbf{x} = e^{\mathrm{i}t} \cdot \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} = (\cos t + \mathrm{i} \sin t) \cdot \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}}_{\mathbf{x}_1} + \mathrm{i} \underbrace{\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}}_{\mathbf{x}_2}.$$

(c) Here, we note that:

$$\mathbf{x}_1' = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix} \text{ and } \mathbf{x}_2' = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix},$$

and we can easily verify that both of them satisfies the differential equation above.
For linear independence, we have the Wronskian as:

$$\mathrm{W}\left[\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}\right] = \det \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \sin^2 t + \cos^2 t = 1 \neq 0,$$

hence $\mathbf{x}_1$ and $\mathbf{x}_2$ forms a fundamental set of linearly independent solutions.