Problem Set 11

Differential Equations

Spring 2025

We hope that your second midterm was smooth and successful. For this week, we will explore deeper into the linear systems as well as special cases in linear system, *i.e.*, complex or zero eigenvalues. Alias, lets explore further on what eigenvalues represent and the geometries behind it.

1. (Linear System versus Second Order). Let an initial value problem for linear system on $x_1 := x_1(t)$ and $x_2 := x_2(t)$ be defined as follows:

$$\begin{cases} x_1' = 3x_1 - 2x_2, & x_1(0) = 3, \\ x_2' = 2x_1 - 2x_2, & x_2(0) = \dfrac{1}{2}. \end{cases}$$

   (a) Solve for the *general solution* for the linear system by considering $\mathbf{x} = (x_1, x_2)$.

   (b) Transform the *general system* into a single equation of second order. Then solve the second-order equation. Eventually, convert your solution of one variable back to the *general solution* to $x_1(t)$ and $x_2(t)$.

   (c) Find the particular solution using the initial conditions, then graph the parameterized curve on a x_1x_2-plane with $t \geq 0$.

2. (A "Big" Matrix). Let $\mathbf{x} = (x_1, x_2)$ satisfy the following differential equation.

$$\mathbf{x}' = \begin{pmatrix} \dfrac{1}{42} & \dfrac{1}{21} \\ \dfrac{1}{14} & \dfrac{1}{21} \end{pmatrix} \cdot \mathbf{x}.$$

   Hint: Think about the *geometric* interpretation of eigenvalues and eigenvectors and try to *simplify* the matrix. (*Otherwise, the computation is hard.*)

3. (Repeated Eigenvalue). This problem investigates the case for repeated eigenvalues. First, we let the matrix $A \in \mathbb{R}^{2\times 2}$ be:

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

   Here, we define the *algebraic multiplicity* of an eigenvalue as its multiplicity as a root to the characteristic polynomial, and the *geometric multiplicity* is the dimension of the eigenspace.

   (a) Find the eigenvalue and its corresponding eigenvector. State their multiplicities.

(b) Find a the general solution to $\mathbf{x}' = A.\mathbf{x}$, where $\mathbf{x} = (x_1, x_2)$.

Then, we consider the diagonal n-by-n matrices, that is matrices with entries only on the diagonal, which can be characterized as:

$$D = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

(c)* Show that the eigenvalues are exactly $a_1, \cdots, a_n$, and the algebraic multiplicity is exactly the same as geometric multiplicity for all eigenvalues.

(d) Consider the linear system $\mathbf{x}' = D \cdot \mathbf{x}$ for $\mathbf{x} = (x_1, \cdots, x_n)$. Explain why do not have to find the eigenvalues in this case.

4. (Zero Eigenvalue). Let a system of $\mathbf{x} = (x_1, x_2)$ be defined as:

$$\mathbf{x}' = \begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} \cdot \mathbf{x}.$$

(a) Find the eigenvalues and eigenvectors for $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix}$.

(b) Give a full set of solutions to the differential equation. Plot some trajectory on the x_1x_2-plane.

(c)* Let A be an arbitrary square matrix. Show that A is non-invertible if and only if A has zero as an eigenvalue.
Note: Please avoid using the definition that the determinant is the product of all eigenvalues. Moreover, consider the geometric implication of eigenvalue to account for invertibility.

Tip of the Week

All Things Go is one of the biggest music festivals in the DC/Baltimore areathe annual fall event is known for its women-centric lineups and support of LGBTQ+ artists, making it a popular weekend trip for Hopkins students every year. The 2025 lineup drops soon, so head to @allthingsgo on IG to see how you can get tickets!