Problem Set 13

Differential Equations

Spring 2025

In the digressions to non-linear system, we are gradually seeing more dynamical systems and applications to the real life situations. As we explore cases with the non-linear system, please keep in mind that we are always trying to find a linear case to model as we zoom in. At the same moment, please take a moment to review what we have learned together.

- Concepts:
  - Nonlinear System
  - Locally Linear System
  - Jacobian Matrix
- Models:
  - Predator-Pray Model
  - Competing Species Model
  - Limit Cycles*

Tip of the Week

As William Shakespeare once said, "The PILOT learning experience fundamentally changes one's life for the better, providing the perfect dichotomy of hands-on learning and peer mentorship in a rigorous academic environment." I hope, over this past semester, you felt the same. Good luck for finals, and, as always, make sure to sleep, eat, and breathe.

1. (System with Unknown Coefficients). Let a non-linear system for $x = x(t)$ and $y = y(t)$ be:

$$\begin{cases} x' = \alpha x - y + y^2, \\ y' = x + \alpha y. \end{cases}$$

   (a) Show that $(0, 0)$ is a critical point, and show system is locally linear at $(0, 0)$ for all $\alpha \in \mathbb{R}$.

   (b) Classify the critical point $(0, 0)$ and sketch a few phase portraits of the linearized system.

2. (Nonlinear at origin). Let the linear system be:

$$\begin{cases} x' = y, \\ y' = x + 2x^3. \end{cases}$$

   (a) Show that the origin is a saddle point.

   (b) Sketch a phase portrait for the linearized system. Note that where all the trajectories of the linear system tend to the origin.

3. (Modeling Politics). Suppose D and R are two parties on a non-existing country on the center of Mars. For the simplicity of this problem, they, *unfortunately*, have no elections. Therefore, we can model the amount of the supporter for each party (in millions), denoted x_D and x_R with the following relationship:

$$\begin{cases} \dfrac{dx_D}{dt} = x_D(1 - x_D - x_R), \\ \dfrac{dx_R}{dt} = x_R(3 - 2x_D - 4x_R). \end{cases}$$

Find all possible endings (say arbitrarily long after, that is $t \to \infty$) of the number of supporters (in millions) for the two parties.

4. (A Case Study on Tariffs). Suppose the tariff system in Mars (between all countries there) is based on the same formula, which is as follows:

$$\Delta\tau_i = \frac{x_i - m_i}{\varepsilon \times \varphi \times m_i}.$$

Here, $\Delta\tau_i$ means the change in tariff, x_i means the total import sale into your country from country i, m_i means the total export sale from your country to the country i, and $\varepsilon \times \varphi$ is 2.
Furthermore, a numerical estimation method in ODEs is called *Euler's Method*, and we will use the reverse of that to obtain an ODE model that:

$$\frac{d\tau(t)}{dt} \approx \frac{x(t) - m(t)}{2m(t)}.$$

Now, suppose there is another county, and you want to analyze the trends of tariffs with that country. With ϑ denoting their country's tariff on your country's import, we can create a system.

$$\begin{cases} \dfrac{d\tau(t)}{dt} = \dfrac{x(t) - m(t)}{2m(t)}, \\ \dfrac{d\vartheta(t)}{dt} = \dfrac{m(t) - x(t)}{2x(t)}. \end{cases}$$

For the simplicity of economics, we can model the import sale and export sale as:

$$x(t) = a - b\tau(t) \qquad \text{and} \qquad m(t) = c - d\vartheta(t),$$

where a, b, c, d are positive real constants.

(a) Write down the system of differential equations to model the tariffs as a vector $\mathbf{x}(t) = \big(\tau(t), \vartheta(t)\big)$.

(b) Find the set of all equilibrium points on this nonlinear system.

(c)* Interpret some issues with the assumptions of this model.

Congratulations for finishing the last weekly PILOT set for the semester.

We wish you the best of all for your finals and your upcoming academic & career pursuits.