Problem Set 4

Differential Equations

Spring 2025

As we wrap up the first part of the class, let's remark on the key components so far about first order ordinary differential equations:

- Concepts:
  - Existence
  - Uniqueness
  - General Solutions
  - Specific Solutions (IVPs)
- Methods to solve first order ODEs:
  - Separable ODEs
  - Integrating Factor
  - Exact ODEs
  - Autonomous ODEs
- Behavior analyses:
  - Directional Field
  - End Behavior
  - Phase Line
  - Bifurcation Diagram

1. (IVP with Specific Domain). Given following IVP for $y := y(x)$.
$$\begin{cases} y' = \dfrac{1}{x^4 - 1}, \\ y(0) = 0. \end{cases}$$
Find the specific solution and state the domain of the solution.

2. (Bifurcation Diagram). For the first-order autonomous ODE:
$$\frac{dx}{dt} = x^2 - 2x + c,$$
with parameter $c \in \mathbb{R}$, do the following:
   (a) Sketch all of the qualitatively different graphs of $f(x) = x^2 - 2x + c$, as c is varied.
   (b) Determine any and all bifurcation values for the parameter c.
   (c) Sketch a bifurcation diagram for this ODE.

3. (Make an Exact ODE). Let a differential equation on $y := y(x)$ be defined as follows:
$$xy^2 + bx^2y + (x + y)x^2y' = 0.$$
Suppose this differential equation is exact. Find the appropriate value of b and then solve for the solution of the differential equation.

4. (Existence and Uniqueness for IVP). Suppose $f(x)$ is non-zero, let an initial value problem be:

$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

(a) Show that the differential equation is **not** linear.

For the next two questions, suppose $f(x) = \tan x$.

(b) State, <u>without</u> justification, the open interval(s) in which $f(x)$ is continuous.

(c)* Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

(d) Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

Tip of the Week

Interested in getting involved in research and don't know where to start? The Hopkins Office of Undergraduate Research (HOUR) connects students with open opportunities based on their interests and area of study. They also manage competitive grants like PURA and the BDP Summer Program. Check out their office at the Imagine Center or click here for more: https://hour.jhu.edu/.