Problem Set 5

Differential Equations

Spring 2025

We hope that your preparation of the first midterm has been going on well. As we wrapped up the first order differential equation, we will examine a few cases with the existence and uniqueness problems in detail, and build up to the foundations of more classes of differential equations.

Tip of the Week

Ready to see the world? The Global Education Office has resources for Study Abroad programs in over 50 countries worldwide, for all areas of study. They provide help with applications, scholarships, and accommodations. Find out more information here https://studyabroad.jhu.edu/!

1. ("Dilemma" with Existence & Uniqueness Theorem). Let a first order IVP on $y := y(t)$ be defined as follows:

$$\begin{cases} y' = \dfrac{2}{t}y, \\ y(1) = 1. \end{cases}$$

   (a) Find the solution to the above initial value problem.

   (b) Recall the theorem on existence and uniqueness, as follows:

For an IVP in simple form: $\begin{cases} \dfrac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$ For some $I = (\alpha, \beta) \ni t_0$, if $a(t)$ and $b(t)$ are continuous on the interval I. Then, there exists a unique solution to the IVP on the interval I.

   Show that the IVP in this problem does not satisfy the condition for the existence and uniqueness theorem for $\mathbb{R}$.

   (c) Does the above example violates the existence and uniqueness theorem? Why?

2. (Some Criterion over intervals). Suppose we have an initial value problem over $y := y(t)$:

$$\begin{cases} y' = F(t, y), \\ y(t_0) = y_0. \end{cases}$$

We suppose that $F(t, y)$ and $\frac{\partial}{\partial y} F(t, y)$ are continuous over a region $I \times J$. Determine if Picard's theorem can guarantee the existence of a uniqueness solution.

(a) $I = (0, 1)$, $J = (0, 2)$, $t_0 = 0.5$, and $y_0 = 1$.

(b) $I = [0, 1]$, $J = [0, 2]$, $t_0 = 0.5$, and $y_0 = 1$.

(c) $I = [0, 1]$, $J = [0, 2]$, $t_0 = 1$, and $y_0 = 1$.

(d) $I = \bigcup_{i=1}^{\infty} [1/i, 1]$, $J = [0, 2]$, $t_0 = 0.5$, and $y_0 = 1$.

(e) $I = \bigcup_{i=1}^{\infty} [1/i, 1]$, $J = [0, 2]$, $t_0 = \delta$, and $y_0 = 1$, where δ is any fixed number on $(0, 1)$.

3. (Existence of Largest Interval). For the following IVPs, determine the largest interval in which a solution is guaranteed to exist.

(a)
$$\begin{cases} (t-3)y' + (\log t)y = 2t, \\ y(1) = 2. \end{cases}$$

(b)
$$\begin{cases} (4-t^2)y' + 2ty = 3t^2, \\ y(1) = -3. \end{cases}$$

(c)
$$\begin{cases} y' + (\tan t)y = \sin t, \\ y(\pi) = 0. \end{cases}$$

4. (Preliminary to Second Order ODEs). Let a second order differential equation be defined as follows:

$$y'' - 2y' + y = 0.$$

(a) Verify that $y_1 = e^t$ and $y_2 = te^t$ are two solutions to the above differential equation.

(b) Verify that any *linear combination* of y_1 and y_2 is a solution to the above differential equation.