Problem Set 5: Solutions

Differential Equations

Spring 2025

1. ("Dilemma" with Existence & Uniqueness Theorem). Let a first order IVP on $y := y(t)$ be defined as follows:
$$\begin{cases} y' = \dfrac{2}{t}y, \\ y(1) = 1. \end{cases}$$
   (a) Find the solution to the above initial value problem.

   (b) Recall the theorem on existence and uniqueness, as follows:

   > For an IVP in simple form:
   > $$\begin{cases} \dfrac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$
   > For some $I = (\alpha, \beta) \ni t_0$, if $a(t)$ and $b(t)$ are continuous on the interval I. Then, there exists a unique solution to the IVP on the interval I.

   Show that the IVP in this problem does not satisfy the condition for the existence and uniqueness theorem for $\mathbb{R}$.

   (c) Does the above example violates the existence and uniqueness theorem? Why?

Solution:

(a) This problem is clearly separable, we may compute:
$$\frac{dy}{y} = 2\frac{dt}{t}$$
$$\int \frac{dy}{y} = 2\int \frac{dt}{t}$$
$$\log|y| = 2\log|t| + C$$
$$y = \tilde{C}t^2.$$
Note that the initial condition enforces that $y(1) = 1$, so the solution is just:
$$y = \boxed{t^2}.$$

(b) Note that $a(t) = 2/t$, which is not continuous over $(-\infty, 0) \cup (0, \infty)$, then the theorem does not guarantee the existence and uniqueness of a solution over $\mathbb{R}$.

(c) This is not a violation since the converse of the theorem is not necessarily true. In propositional logic, if A implies B (written as $A \implies B$), the converse (B implies A, written as $B \implies A$) is not necessarily true. Hence, we can still have a solution that is unique over $\mathbb{R}$.

2. (Some Criterion over intervals). Suppose we have an initial value problem over $y := y(t)$:

$$\begin{cases} y' = F(t,y), \\ y(t_0) = y_0. \end{cases}$$

We suppose that $F(t,y)$ and $\frac{\partial}{\partial y}F(t,y)$ are continuous over a region $I \times J$. Determine if Picard's theorem can guarantee the existence of a uniqueness solution.

(a) $I = (0,1)$, $J = (0,2)$, $t_0 = 0.5$, and $y_0 = 1$.

(b) $I = [0,1]$, $J = [0,2]$, $t_0 = 0.5$, and $y_0 = 1$.

(c) $I = [0,1]$, $J = [0,2]$, $t_0 = 1$, and $y_0 = 1$.

(d) $I = \bigcup_{i=1}^{\infty}[1/i, 1]$, $J = [0,2]$, $t_0 = 0.5$, and $y_0 = 1$.

(e) $I = \bigcup_{i=1}^{\infty}[1/i, 1]$, $J = [0,2]$, $t_0 = \delta$, and $y_0 = 1$, where δ is any fixed number on $(0,1)$.

Solution:

(a) Yes, since $(t_0, y_0) \in I \times J$ and there exists $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset I$.

(b) Yes, since $(t_0, y_0) \in I \times J$ and there exists $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset I$.

(c) No, although $(t_0, y_0) \in I \times J$ but there does not exist any $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset I$.

For parts (d) and (e), $I = (0,1]$.

(d) Yes, since $(t_0, y_0) \in I \times J$ and there exists $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset I$.

(e) Yes, since $(\delta, y_0) \in I \times J$ and there exists $\delta' = \delta/2 > 0$ such that $(\delta - \delta', \delta + \delta) \subset I$.

Note that this question includes various interesting topics in *point-set topology*, such as:

- Open and closed sets in relation to interior points and boundary points,
- Finite/countable union of open/closed sets, or
- Basic σ-algebra and Borel set.

Look into these topics if you find them interesting.

3. (Existence of Largest Interval). For the following IVPs, determine the largest interval in which a solution is guaranteed to exist.

(a)
$$\begin{cases} (t-3)y' + (\log t)y = 2t, \\ y(1) = 2. \end{cases}$$

(b)
$$\begin{cases} (4-t^2)y' + 2ty = 3t^2, \\ y(1) = -3. \end{cases}$$

(c)
$$\begin{cases} y' + (\tan t)y = \sin t, \\ y(\pi) = 0. \end{cases}$$

Solution:

(a) In standard form, we have:
$$y' = -\frac{\log t}{t-3}y + \frac{2t}{t-3}.$$
We note discontinuities at $t = 0$ and $t = 3$, so the interval such that a unique solution exists guaranteed by the theorem is $\boxed{(0,3)}$.

(b) In standard form, we have:
$$y' = \frac{2t}{(t+2)(t-2)}y + \frac{3t^2}{(2+t)(2-t)}.$$
We note discontinuities at $t = -2$ and $t = 2$, so the interval such that a unique solution exists guaranteed by the theorem is $\boxed{(-2,2)}$.

(c) In standard form, we have:
$$y' = -\frac{\sin t}{\cos t}y + \sin t.$$
We note discontinuities at $t = (2k+1)\pi/2$ for $k \in \mathbb{Z}$, so the interval such that a unique solution exists guaranteed by the theorem is $\boxed{(\pi/2, 3\pi/2)}$.

4. (Preliminary to Second Order ODEs). Let a second order differential equation be defined as follows:

$$y'' - 2y' + y = 0.$$

(a) Verify that $y_1 = e^t$ and $y_2 = te^t$ are two solutions to the above differential equation.

(b) Verify that any *linear combination* of y_1 and y_2 is a solution to the above differential equation.

Solution:

(a) Clearly, we can check that:

$$\begin{aligned} y_1' &= e^t, & y_1'' &= e^t, \\ y_2' &= te^t + e^t & y_2'' &= te^t + 2e^t. \end{aligned}$$

Then, as we plug them into the left hand side of the differential equation, we have:

$$e^t - 2e^t + e^t = 0 \text{ and } te^t + 2e^t - 2(te^t + e^t) + te^t = 0.$$

Hence, they are both solutions to the second order differential equation.

(b) Clearly, if y_1 and y_2 are solutions to the above differential equation, we have:

$$\begin{cases} y_1'' - 2y_1' + y_1 = 0, \\ y_2'' - 2y_2' + y_2 = 0. \end{cases}$$

Hence, adding the above equations with multiples of λ_1 and λ_2, such that $\lambda_1, \lambda_2 \in \mathbb{R}$, and use the linearity of derivative operator gives that:

$$\lambda_1(y_1'' - 2y_1' + y_1) + \lambda_2(y_2'' - 2y_2' + y_2) = (\lambda_1 y_1 + \lambda_2 y_2)'' - 2(\lambda_1 y_1 + \lambda_2 y_2)' + (\lambda_1 y_1 + \lambda_2 y_2) = 0.$$

Therefore, $\lambda_1 y_1 + \lambda_2 y_2$ is still a solution to the above differential equation.