PILOT

Problem Set 9

Differential Equations

Spring 2025

As of right now, we have completed our expenditure of higher order differential equations. You should be familiar with the following concepts:

- Concepts:
  - Set of Solutions
  - Linear Independence
  - Existence & Uniqueness Theorem
- Methods to solve higher order ODEs:
  - Characteristic Equation
  - Euler's Formula
  - Undetermined Coefficients
  - Reduction of Order
  - Variation of Parameters

Now, as we step into more linear algebra, we are going to review the key contents of this part of the class.

Tip of the Week

Fall 2025 registration is coming up soon! Make sure to consult with your academic and faculty advisors, check the e-catalog, and speak with upperclassmen for course recommendations. This year's registration dates are April 7 for rising seniors, April 9 for rising juniors, and April 11 for rising sophomores.

1. (Reduction of Order or Integrating Method). Let a differential equation be:

$$y''(t) + \frac{2}{t}y'(t) = 0.$$

   (a) Verify that $y(t) = 1/t$ is one solution, then find a full set of solution.

   (b) Consider $\omega(t) = y'(t)$, solve the differential equation by using integrating factor.

   (c) Verify that the two methods give you the same set of the solutions.

2. (Non-homogeneous Cases of Higher Order ODEs). Let a third order differential equation be as follows:

$$\ell[y(t)] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$

Let $\ell[y(t)] = 0$ be trivial initially.

(a) Find the set of all linearly independent solutions.

Then, assume that $\ell[y(t)]$ is non-trivial.

(b) Find the particular solution to $\ell[y(t)] = \sin t$.

(c) Find the particular solution to $\ell[y(t)] = e^{-t}$.

(d)* Suppose that $\ell[y_1(t)] = f(t)$ and $\ell[y_2(t)] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell[y_3(t)] = f(t) + g(t)$.

3. (Non-homogeneous Differential Equations). Solve the following differential equations.

   (a) $$y'' + 4y = t^2 + 3e^t.$$

   (b) $$y'' + 2y' + y = \frac{e^{-x}}{x}.$$

4. (Warm up in Linear Algebra). This problem reviews the basic concepts linear algebra concepts.

   (a) Which of the following set of vectors are linearly independent in $\mathbb{R}$-vector space, what about $\mathbb{C}$-vector space? Justify your answer.

   (i) $$\alpha = \{(1,1,0),(0,1,1),(1,0,1)\},$$

   (ii) $$\beta = \{(0,1),(2,3),(4,5)\},$$

   (iii) $$\gamma = \{1,\mathrm{i}\}.$$

   (b) Let $A = \begin{pmatrix} 1+\mathrm{i} & -1+2\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{pmatrix}$ and $B = \begin{pmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{pmatrix}$, compute the following:

   (i) $$A - 2B,$$

   (ii) $$BA,$$

   (iii) $$B^{-1}.$$