Midterm 1 Review Problem Set: Solutions

Differential Equations

Spring 2025

1. Find the general solution for $y = y(t)$:

$$y' + 3y = t + e^{-2t},$$

then, describe the behavior of the solution as $t \to \infty$.

Solution:

Here, one could note that this differential equation is not separable but in the form of integrating factor problem, then we find the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t 3ds\right) = \exp(3t).$$

By multiplying both sides with $\exp(3t)$, we obtain the equation:

$$y'e^{3t} + 3ye^{3t} = te^{3t} + e^{-2t}e^{3t}.$$

Clearly, we observe that the left hand side is the derivative after product rule for ye^{3t} and the right hand side can be simplified as:

$$\frac{d}{dt}[ye^{3t}] = te^{3t} + e^t.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$\begin{aligned} ye^{3t} &= \int te^{3t}dt + \int e^t dt \\ &= \frac{te^{3t}}{3} - \int \frac{1}{3}e^{3t}dt + e^t + C \\ &= \frac{te^{3t}}{3} - \frac{e^{3t}}{9} + e^t + C. \end{aligned}$$

Eventually, we divide both sides by e^{3t} to obtain that:

$$y(t) = \boxed{\frac{t}{3} - \frac{1}{9} + e^{-2t} + Ce^{-3t}}.$$

As $t \to \infty$, the solution diverges to ∞.

2. Given an initial value problem:

$$\begin{cases} \dfrac{dy}{dt} - \dfrac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$

   (a) Find the integrating factor $\mu(t)$.

   (b) Solve for the particular solution for the initial value problem.

   (c) Discuss the behavior of the solution as $t \to \infty$ for different cases of y_0.

Solution:

(a) As instructed, we look for the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t -\frac{3}{2}ds\right) = \boxed{\exp\left(-\frac{3}{2}t\right)}.$$

(b) With the integrating factor, we multiply both sides by $\mu(t)$ to obtain that:

$$y'e^{-3t/2} - \frac{3}{2}ye^{-3t/2} = 3te^{-3t/2} + 2e^te^{-3t/2}.$$

Clearly, we observe that the left hand side is the derivative after product rule for $ye^{-3t/2}$ and the right hand side can be simplified as:

$$\frac{d}{dt}\left[ye^{-3t/2}\right] = 3te^{-3t/2} + 2e^{-t/2}.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$\begin{aligned} ye^{-3t/2} &= \int 3te^{-3t/2}dt + \int 2e^{-t/2}dt \\ &= -2te^{-3t/2} + 2\int e^{-3t/2}dt - 4r^{-t/2} + C \\ &= -2te^{-3t/2} - \frac{4}{3}e^{-3t/2} - 4r^{-t/2} + C. \end{aligned}$$

Then, we divide both sides by $e^{-3t/2}$ to get the general solution:

$$y(t) = -2t - \frac{4}{3} - 4e^t + Ce^{3t/2}.$$

Given the initial condition, we have that:

$$y_0 = 0 - \frac{4}{3} - 4 + C,$$

which implies $C = 16/3 + y_0$, leading to the particular solution that:

$$y(t) = \boxed{-2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}}.$$

(c) We observe that:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[-2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}\right].$$

Note that the important terms are e^t and $e^{3t/2}$, we need to care the critical value $-16/3$:

- when $y_0 > -16/3$, $\boxed{y(t) \to \infty \text{ when } t \to \infty}$,
- when $y_0 \leq -16/3$, $\boxed{y(t) \to -\infty \text{ when } t \to \infty}$.

3. An autonomous differential equation is given as follows:

$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \text{ where } t \geq 0 \text{ and } y \geq 0.$$

Draw a phase portrait and sketch a few solutions with different initial conditions.

Solution:

Recall the *Rational root test* (c.f. §1.3). Let the polynomial with integer coefficients be defined as:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p|a_0$ and $q|a_n$. From the theorem, we can note that if the equation has a rational root, it must be one of:

$$r = \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{4}.$$

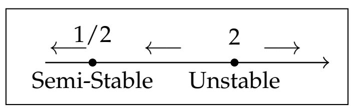
By plugging in, one should notice that $y = 2$ is a root (one might also notice $1/2$ is a root as well, but we will get the step slowly), so we can apply the long division (dividing $y - 2$) to obtain that:

$$\frac{4y^3 - 12y^2 + 9y - 2}{y - 2} = 4y^2 - 4y + 1.$$

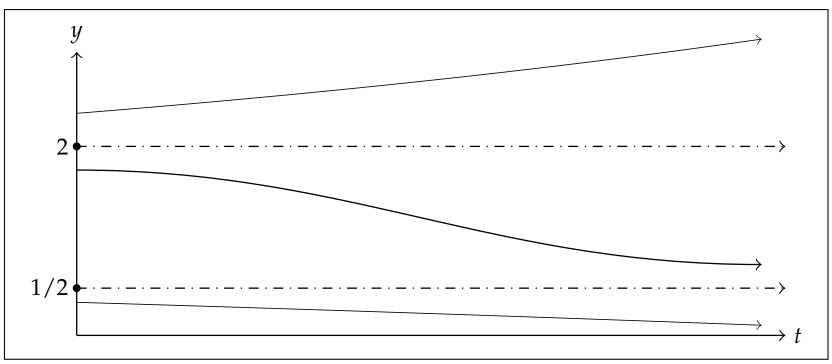
Clear, we can notice that the right hand side is a perfect square (else, you could use the quadratic formula) that:

$$4y^2 - 4y + 1 = (2y - 1)^2.$$

Thus, we now know that the roots are 2 and $1/2$ (multiplicity 2). Hence, the phase portrait is:



Correspondingly, we can sketch a few solutions (not necessarily in scale):



4. Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:

$$y'(x) = e^{2x} + y(x) - 1.$$

Solution:

First, we write the equation in the general form:

$$\frac{dy}{dx} + (1 - e^{2x} - y) = 0.$$

Now, we take the partial derivatives to obtain that:

$$\frac{\partial}{\partial y}[1 - e^{2x} - y] = -1,$$

$$\frac{\partial}{\partial x}[1] = 0.$$

Notice that the mixed partials are not the same, the equation is $\boxed{\text{not exact}}$.

Here, we choose the integrating factor as:

$$\mu(x) = \exp\left(\int_0^x \frac{\frac{\partial}{\partial y}[1 - e^{2s} - y] - \frac{\partial}{\partial s}[1]}{1} ds\right)$$

$$= \exp\left(\int_0^x -ds\right) = \exp(-x).$$

Therefore, our equation becomes:

$$(e^{-x})\frac{dy}{dx} + (e^{-x} - e^{x} - ye^{-x}) = 0.$$

After multiplying the integrating factor, it would be exact. *We leave the repetitive check as an exercise to the readers*.

Now, we can integrate to find the solution with a $h(y)$ as function:

$$\varphi(x, y) = \int (e^{-x} - e^{x} - ye^{-x})dx = -e^{-x} - e^{x} + ye^{-x} + h(y).$$

By taking the partial derivative with respect to y, we have:

$$\partial_y \varphi(x, y) = e^{-x} + h'(y),$$

which leads to the conclusion that $h'(y) = 0$ so $h(y) = C$.

Then, we can conclude that the solution is now:

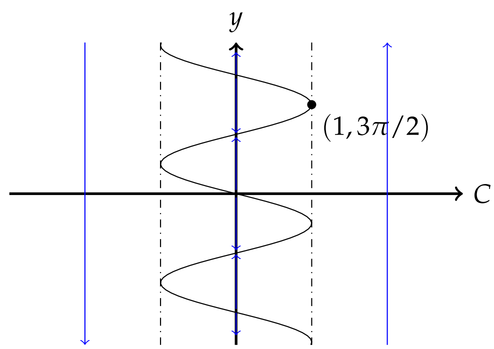
$$\varphi(x, y) = -e^{-x} - e^{x} + ye^{-x} + C = 0,$$

which is equivalently:

$$y(x) = \boxed{\widetilde{C}e^{x} + 1 + e^{2x}}.$$

5. For the first-order autonomous ODE:
$$\frac{dy}{dt} = \sin y + C,$$
where $C \in \mathbb{R}$ is a parameter. Determine any and all bifurcation values for the parameter C and sketch a bifurcation diagram.

Solution:
It is not hard to observe that $\sin y$ will intersect the axis infinitely many times, while $\sin(\mathbb{R}) = [-1, 1]$, one shall then realize that the bifurcation value would be ± 1, since when $C > 1$ or $C < -1$, there will be no equilibriums at all.
Therefore, the bifurcation diagram can be illustrated as:



6. Let an initial value problem be defined as follows:

$$\begin{cases} (12x^4 + 5x^2 + 6)\dfrac{dy}{dx} - (x^2\sin(x) + x^3)y = 0, \\ y(0) = 1. \end{cases}$$

Show that the solution to the above initial value problem is symmetric about $x = 0$.

Solution:

If you were attempting to solve this problem by integrating factor or exactness, you are on the wrong track. The functions are deliberately chosen so that these operations will be hardly possible.

However, this does not necessarily mean that is it not possible to prove without solving the solution out, one shall utilize the existence and uniqueness theorem to proceed.

Proof. Now, we first observe that when we rewrite the problem, we have:

$$y' = \frac{x^2\sin(x) + x^3}{12x^4 + 5x^2 + 6}y,$$

where we clearly notice that the numerator and denominator are composed of continuous function while the denominator is positive, so we know that it is continuous over $\mathbb{R}$, so the initial value problem exhibits a unique solution.

Now, suppose $y(x)$ is a solution of the above IVP, we want to show that $\tilde{y}(x) := y(-x)$ is also a solution to the above IVP.

Clearly, we have:

$$\tilde{y}(0) = y(-0) = y(0) = 1,$$

so the initial condition is satisfied, so we are left to check the differential equation. By chain rule, we have:

$$\frac{d\tilde{y}}{dx}(x) = \frac{dy}{dx}(-x) \cdot \frac{d}{dx}[-x] = -\tilde{y}'(x).$$

With the first equation and y being a solution, we can make all x into $-x$ to obtain that:

$$(12(-x)^4 + 5(-x)^2 + 6)\frac{dy(-x)}{dx} - ((-x)^2\sin(-x) + (-x)^3)\tilde{y} = 0,$$

and if we organize the left hand side, we have:

$$(12x^4 + 5x^2 + 6)\tilde{y}' - (x^2\sin(x) + x^3)\tilde{y} = 0,$$

so $\tilde{y}$ is clearly another solution to the IVP, so by uniqueness, we must have $\tilde{y}(x) = y(x)$, or namely $y(-x) = y(x)$, so the solution must be symmetric about $x = 0$. $\square$