Final Review Problem Set: Solutions

Differential Equations

Spring 2025

1. Let systems of differential equations be defined as follows, find the general solutions to the equations:

(a) $$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = (x_1, x_2).$$

(b) $$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}, \qquad \mathbf{x} = (x_1, x_2).$$

(c) $$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = (x_1, x_2, x_3).$$

Solution:

(a) Here, we notice that the liner system is diagonal, so we can simply solve for each entry as $x_1 = e^{3t}$ and $x_2 = e^{2t}$. Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

(b) We first solve for the eigenvalues and eigenvectors:

$$0 = \det \begin{pmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{pmatrix} = (3 - \lambda)(-1 - \lambda) + 8 = \lambda^2 - 2\lambda + 5.$$

Hence, the eigenvalues are $\lambda = 1 \pm 2\mathrm{i}$. Since they are complex conjugates, we pick $\lambda_1 = 1 - 2i$, so $\boldsymbol{\xi}^{(1)}$ satisfies $\begin{pmatrix} 2 + 2\mathrm{i} & -2 \\ 4 & -2 + 2\mathrm{i} \end{pmatrix} . \xi^{(1)} = \mathbf{0}$, so we have $(2 + 2i)\xi_1^{(1)} = 2\xi_2^{(1)}$, so we the eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1 + \mathrm{i})$. We can get our solution:

$$\mathbf{x} = e^{(1-2i)t} \begin{pmatrix} 1 + \mathrm{i} \\ 1 \end{pmatrix} = e^t \left(\cos(2t) - \mathrm{i} \sin(2t) \right) \begin{pmatrix} 1 \\ 1 + \mathrm{i} \end{pmatrix}$$

$$= e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + \mathrm{i} e^t \begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + C_2 e^t \begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}}.$$

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(c) Again, we first find the eigenvalues of the equation, *i.e.*:

$$\det \begin{pmatrix} 1-\lambda & 0 & 4 \\ 1 & 1-\lambda & 3 \\ 0 & 4 & 1-\lambda \end{pmatrix} = 0,$$

which is $(1-\lambda)^3 + 16 - 12(1-\lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = -(\lambda+1)^2(\lambda-5) = 0$. Hence, the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Now, we look for eigenvectors.

- For $\lambda_1 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $x \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$.
- For $\lambda_2 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} 4x \\ x+1 \\ -2x-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
- For $\lambda_3 = 5$, we have $\begin{pmatrix} -4 & 0 & 4 \\ 1 & -4 & 3 \\ 0 & 4 & -4 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$, which is $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + C_2 \left(t e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) + C_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}.$$

2. Solve the following initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution:
Here, we first find the eigenvalues for the matrix, that is:

$$\det \begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} = 0.$$

Therefore, the polynomial is $(1-\lambda)(-7-\lambda)+16 = (\lambda+3)^2 = 0$, hence the eigenvalues is $\lambda_1 = \lambda_2 = -3$. Then, we look for the eigenvectors.

- For $\lambda_1 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $\boldsymbol{\xi}_1 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} x \\ x-1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$.

Hence, the general solution is:

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} \right).$$

By the initial condition, we have $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, so:

$$\mathbf{x}(0) = \begin{pmatrix} C_1 + 0 \\ C_1 - C_2/4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore, $C_1 = 3$ and $C_2 = 4$, so the particular solution is:

$$\mathbf{x}(t) = \boxed{\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} te^{-3t}}.$$

3. For the following non-linear systems, find all equilibrium(s) and classify their stability locally if they are locally linear.

(a)
$$\begin{cases} \dfrac{dx}{dt} = x - y^2, \\ \dfrac{dy}{dt} = x + x^2 - 2y. \end{cases}$$

(b)
$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y^2, \\ \dfrac{dy}{dt} = x + 4y^2. \end{cases}$$

Solution:

(a) For the first case, we notice that the equilibrium points are if:
$$\begin{cases} x - y^2 = 0, \\ x + x^2 - 2y = 0. \end{cases}$$
Note that this will be two parabolas, and there are at most two intersections, and we observe the intersections $(0,0)$ and $(1,1)$. Also to note, the Jacobian matrix is:
$$\mathrm{J} = \begin{pmatrix} 1 & -2y \\ 1+2x & -2 \end{pmatrix}.$$

- For the $(0,0)$ case, we denoting $\mathbf{x} = (x, y)$, we verify the linear approximation as:
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x},$$
and we note that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, and by:
$$\lambda_2 < 0 < \lambda_1,$$
we know that we have a unstable saddle point at $(0,0)$.
- For the $(1,1)$ case, we have the linear approximation as:
$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix},$$
and we note the eigenvalues are $\lambda = \frac{-1 \pm i\sqrt{15}}{2}$, which is complex with a negative real part, so we have a $\boxed{\text{asymptotically stable spiral point}}$.

(b) Here, we note that the equilibrium(s) is achieved if and only if $x' = y' = 0$, that is:
$$\begin{cases} 2x + 3y^2 = 0, \\ x + 4y^2 = 0. \end{cases}$$
In particular, we consider $z = y^2$, so we have a system of linear equations, which simplifies to $x = y = 0$, hence the only equilibrium is at $(x, y) = (0, 0)$.
Then, we consider the system locally, denoting $\mathbf{x} = (x, y)$, that is:
$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x},$$
and note that the determinant of the Jacobian matrix is zero, so it is not locally linear, so we cannot conclude any information from this.

4. Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:

$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$

Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.

(a) Find the set of all equilibrium(s) for $\mathbf{x}$.

(b) Find the set in which the equilibrium(s) is locally linear.

Now, $F : \mathbb{R}^2 \to \mathbb{R}$ is not necessarily well-behaved.

(c) Construct a function F such that $\mathbf{x}$ has a equilibrium that is <u>not</u> locally linear.
Hint: Consider the condition in which a non-linear system is locally linear.

Solution:

(a) Here, we note that the equilibrium is when $F(\mathbf{x}) = 0$, *i.e.*, $\sin x_1 + \csc(3x_2) = 0$. Here, we note that the image of $\sin x_1$ is $[-1, 1]$ and the image of $\sec(3x_2)$ is $(-\infty, -1] \sqcup [1, \infty)$, this implies that $\sin x_1 + \sec(3x_2)$ is zero only if $\sin x_1 = \pm 1$ and $\sec(3x_2) = \mp 1$, correspondingly.
First, we consider the set in which x_1 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Correspondingly, we consider the set in which x_2 is -1, that is:

$$\left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Then, we consider the set in which x_1 is -1, that is:

$$\left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Likewise, we consider the set in which x_2 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Therefore, set theoretically, we have the set of all equilibriums as:

$$\boxed{\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}}.$$

(b) Here, one should notice that the Jacobian matrix would have the first row identical with the second row, so its determinant is consistently zero, and it is not locally linear at any value.

(c) Clearly, we must enforce that $F(\mathbf{x})$ is $\boxed{\text{not twice differentiable}}$ with some partial derivatives near the equilibrium point(s). One trivial example could be using the absolute value, such as $F(\mathbf{x}) = |x_1| + |x_2|$, where $(0, 0)$ is a equilibrium but it is <u>not differentiable</u>.
For capable readers, we invite them to look for more functions, such as the Weierstrass Function, a continuous function that is *nowhere* differentiable:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cos(3^k x).$$

5. Let the following systems of (x, y) be functions of variable t:

(a)
$$\begin{cases} x' = (1+x)\sin y, \\ y' = 1 - x - \cos y. \end{cases}$$

(b)
$$\begin{cases} x' = x - y, \\ y' = x - 2y + x^2. \end{cases}$$

Identify the corresponding linear system, then evaluate the stability for the equilibrium at $(0,0)$ by showing it is locally linear.

Solution:

(a) We evaluate x and y both at 0 for the differential equation, and $x' = y' = 0$, so $(0,0)$ is a equilibrium. Then, we can find the Jacobian Matrix:

$$\mathrm{J} = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix},$$

and this implies that the linear system is:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}.$$

As we evaluate J at $(0,0)$ and take its determinant, we have:

$$\det\left(\mathrm{J}\big|_{(0,0)}\right) = \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 \neq 0.$$

Hence, the $(0,0)$ is locally linear.

Here, we have the eigenvalues as $\lambda^2 + 1 = 0$, so they are purely imaginary, so we have an $\boxed{\text{indeterminate spiral or center point}}$.

(b) We evaluate x and y both at 0 for the differential equation, and $x' = y' = 0$, so $(0,0)$ is a equilibrium. Then, we consider the Jacobian matrix as:

$$\mathrm{J} = \begin{pmatrix} 1 & -1 \\ 2x+1 & -2 \end{pmatrix}.$$

Now, we evaluate the matrix at $(0,0)$ and take its determinant:

$$\det(\mathrm{J}\big|_{(0,0)}) = \det \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} = -1 \neq 0.$$

Hence, the system is locally linear, and the linear system locally at $(0,0)$ should be:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

We find its eigenvalue as:

$$0 = \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda) + 1 = \lambda^2 + \lambda - 1.$$

By using the quadratic formula, we have the eigenvalues as $\lambda = \frac{-1 \pm \sqrt{5}}{2}$.

Thus, we have $\lambda_1 < 0 < \lambda_2$, so we have a $\boxed{\text{unstable saddle point}}$.

6. Determine the periodic solution, if there are any, of the following system:

$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \\ y' = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \end{cases}$$

Solution:

Here, we recall the formula converting between polar coordinates and Cartesian coordinates:

$$\begin{cases} x = r\cos\theta, & y = r\sin\theta, \\ rr' = xx' + yy', & r^2\theta' = xy' - yx'. \end{cases}$$

Now, we are able to convert the system as:

$$\begin{cases} rr' = x\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] + y\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right], \\ r^2\theta' = x\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] - y\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right]. \end{cases}$$

Here, by simple deductions, we trivially have:

$$\begin{aligned} rr' &= \frac{x^2+y^2}{\sqrt{x^2+y^2}}(x^2+y^2-2) = \frac{r^2}{r}(r^2-2) \rightsquigarrow r' = r^2-2. \\ r^2\theta' &= -x^2-y^2 = -r^2 \rightsquigarrow \theta' = -1. \end{aligned}$$

Thereby, we consider the radius as:

$$r' = r^2 - 2 = (r-\sqrt{2})(r+\sqrt{2}).$$

Hence, we note that the critical point is $\boxed{r = \sqrt{2}}$ (since r must be positive). Note that $r' < 0$ for $0 < r < \sqrt{2}$ and $r' > 0$ for $r > \sqrt{2}$. Hence, this is an unstable limit cycle.