# Spring Break Special Problem Set

Differential Equations

Spring 2025

We hope that you are enjoying your Spring Break after another intense half of the semester. While you are enjoying your "free" times during the break, here is a problem set for you to work through some interesting problems.

- The questions in this set is more challenging compared to ordinary sets.
- Some questions requires higher math maturity than typical standard for this class.

Have fun, and have a nice Spring Break.

1. (Hilbert Space of Functions). Recall that we have defined a "vector space" of functions for $L^2([0, 2\pi])$ (cf. §6.3). In fact, this is how Fourier series is being defined. Here, $\{\sin(nx), \cos(nx)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis of $L^2([0, 2\pi])$ space.

   (a) Verify that $\{\sin(nx), \cos(nx), 1\}_{n\in\mathbb{Z}^+}$ is an orthogonal set.

   Note that the verification of it being a basis is, in fact, much more complicated, so we will just bear with that. However, for any function $f \in L^2([0, 2\pi])$, it is defined such that:

$$\int_0^{2\pi} \left(f(x)\right)^2 dx < +\infty.$$

   (b) Verify that $f(x) = x$ is a $L^2([0, 2\pi])$ function.

   (c) Decompose $f(x) = x$ into sine and cosine functions, this is a Fourier series of $f(x) = x$.

2. (PDEs: Wave Equation). The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length $L$, the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x, t)$:

$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x, 0) = \sin\left(\dfrac{2\pi x}{L}\right), & \\ & \dfrac{\partial u}{\partial t}(x, 0) = \sin\left(\dfrac{6\pi x}{L}\right), & \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0, t) = u(L, t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$

   where $c$ is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*, $u(x, t) = v(x) \cdot w(t)$, and attempt to obtain a general solution that is *non-trivial*.

   *Hint:* Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L), 1\}_{n\in\mathbb{Z}^+}$ forms an orthogonal basis (Question 1).

3. (*Putnam 2023*: A Linear System). Determine the smallest positive real number $r$ such that there exists differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying:

   - $f(0) > 0$,
   - $g(0) = 0$,
   - $|f'(x)| \leq |g(x)|$ for all $x$,
   - $|g'(x)| \leq |f(x)|$ for all $x$, and
   - $f(r) = 0$.

   You may give an answer *without* a rigorous proof, as the proof is out of scope of the course.
   *Hint:* Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating $f$ and $g$.

4. (Nilpotent Operator). Let $M$ be a square matrix, $M$ is *nilpotent* if $M^k = 0$ for some positive integer $k$. Similar to how we defined the exponential function analytically, the exponential function is also defined for matrices, let $A$ be a square matrix, we define:
$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i.$$

   (a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

   Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

   (b) Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
   *Hint:* Use the differences of squares for matrices.

   (c) If all the entries in $N$ are rational, show that $\exp(N)$ has rational entries.

5. (Rotational Matrix). Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle $\theta$ (counter-clockwise), then:
$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

   (a) Show that $M^\intercal = M^{-1}$.

   (b) Let $\theta = 2\pi/k$ be fixed, where $k$ is an integer. Find the least positive integer $n$ such that $M^n = \mathrm{Id}_2$. Here, $n$ is called the *order* of $M$.
   *Hint:* Consider the rotational matrix geometrically, rather than arithmetically.

   (c) Let $\theta = \pi/2$, calculate the matrix exponential $\exp(M)$.
   *Hint:* Consider the *order* of $M$ and the Taylor series of $e^x$, $e^{-x}$, $\sin x$ and $\cos x$ (cf. §1.2).