Spring Break Special Problem Set: Solutions

Differential Equations

Spring 2025

1. (Hilbert Space of Functions). Recall that we have defined a "vector space" of functions for $L^2([0, 2\pi])$ (cf. §6.3). In fact, this is how Fourier series is being defined. Here, $\{\sin(nx), \cos(nx)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis of $L^2([0, 2\pi])$ space.

   (a) Verify that $\{\sin(nx), \cos(nx), 1\}_{n\in\mathbb{Z}^+}$ is an orthogonal set.

   Note that the verification of it being a basis is, in fact, much more complicated, so we will just bear with that. However, for any function $f \in L^2([0, 2\pi])$, it is defined such that:

$$\int_0^{2\pi} (f(x))^2 dx < +\infty.$$

   (b) Verify that $f(x) = x$ is a $L^2([0, 2\pi])$ function.

   (c) Decompose $f(x) = x$ into sine and cosine functions, this is a Fourier series of $f(x) = x$.

Solution:

(a) We just need to discuss a few cases separately.

- Case for $\sin(nx)$ and $\sin(mx)$, where $n \neq m$:

$$\begin{aligned}\int_0^{2\pi} \sin(nx)\sin(mx)\,dx &= \frac{1}{2}\int_0^{2\pi} [\cos((n-m)x) - \cos((n+m)x)]\,dx \\ &= \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x) - \frac{1}{n+m}\sin((n+m)x)\right]_{x=0}^{x=2\pi} = 0.\end{aligned}$$

- Case for $\sin(nx)$ and $\cos(mx)$:

$$\begin{aligned}\int_0^{2\pi} \sin(nx)\cos(mx)\,dx &= \frac{1}{2}\int_0^{2\pi} [\sin((n+m)x) + \sin((n-m)x)]\,dx \\ &= \frac{1}{2}\left[-\frac{1}{n+m}\cos((n+m)x) - \frac{1}{n-m}\cos((n-m)x)\right]_{x=0}^{x=2\pi} = 0.\end{aligned}$$

- Case for $\cos(nx)$ and $\cos(mx)$, where $n \neq m$:

$$\begin{aligned}\int_0^{2\pi} \cos(nx)\cos(mx)\,dx &= \frac{1}{2}\int_0^{2\pi} [\cos((n-m)x) + \cos((n+m)x)]\,dx \\ &= \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x) + \frac{1}{n+m}\sin((n+m)x)\right]_{x=0}^{x=2\pi} = 0.\end{aligned}$$

Hence, we have verified that this is a orthogonal set. Note that if you want an orthonormal set, you can easily normalize it via:

$$\left\{\frac{\sin(nx/2)}{\sqrt{\pi}}, \frac{\cos(nx/2)}{\sqrt{\pi}}\right\}_{n\in\mathbb{Z}^+}.$$

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This is because:

- Case for $\sin(nx)$ and $\sin(nx)$:
$$\int_0^{2\pi} \sin(nx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi}[1-\cos(2nx)]\,dx = \frac{1}{2}\left[x - \frac{1}{2n}\sin(2nx)\right]_{x=0}^{x=2\pi} = \pi.$$
- Case for $\cos(nx)$ and $\cos(nx)$:
$$\int_0^{2\pi} \cos(nx)\cos(nx)\,dx = \frac{1}{2}\int_0^{2\pi}[1+\cos(2nx)]\,dx = \frac{1}{2}\left[x + \frac{1}{2n}\sin(2nx)\right]_{x=0}^{x=2\pi} = \pi.$$

In fact, one can prove that this is in fact a basis, *i.e.*, it spans the whole $L^2([0,2\pi])$ space. There have been proofs by using Poisson kernel, complex analysis, or by the convergence of Fourier series. We will leave this as an interests for who might get interested in this.

(b) The verification that $f(x) = x$ is trivial:
$$\int_0^{2\pi} (f(x))^2 dx = \int_0^{2\pi} x^2 dx = \frac{1}{3}x^3\Big|_{x=0}^{x=2\pi} = \frac{1}{3}\cdot 8\pi = \frac{8}{3}\pi < +\infty.$$
Hence, we have verified that $f(x) = x$ is $L^2([0,2\pi])$.

(c) Note that we have the orthogonal basis, so we can think about project $f(x) = x$ to each basis:

- Project x to $\sin(nx)$ for $n \in \mathbb{Z}^+$:
$$\begin{aligned}\mathrm{proj}_{\sin(nx)}(x) &= \frac{\langle \sin(nx), x\rangle}{\langle \sin(nx), \sin(nx)\rangle}\sin(nx)\\ &= \frac{\int_0^{2\pi} x\sin(nx)dx}{\pi}\sin(nx)\\ &= \left[-\frac{1}{n}x\cos(nx) + \frac{1}{n}\int \cos(nx)\,dx\right]_{x=0}^{x=2\pi}\frac{\sin(nx)}{\pi}\\ &= \left[-\frac{2\pi}{n}\cos(2n\pi) + 0\right]\cdot\frac{1}{\pi}\sin(nx) = -\frac{2}{n}\sin(nx).\end{aligned}$$
- Project x to $\cos(nx)$ for $n \in \mathbb{Z}^+$:
$$\begin{aligned}\mathrm{proj}_{\cos(nx)}(x) &= \frac{\langle \cos(nx), x\rangle}{\langle \cos(nx), \cos(nx)\rangle}\cos(nx)\\ &= \frac{\int_0^{2\pi} x\cos(nx)dx}{\pi}\cos(nx)\\ &= \left[\frac{1}{n}x\sin(nx) - \frac{1}{n}\int \sin(nx)\,dx\right]_{x=0}^{x=2\pi}\frac{\cos(nx)}{\pi} = 0.\end{aligned}$$
- Project x to 1:
$$\mathrm{proj}_1(x) = \frac{\langle 1, x\rangle}{\langle 1, 1\rangle}\cdot 1 = \frac{\int_0^{2\pi} x dx}{\int_0^{2\pi} 1 dx} = \frac{\frac{x^2}{2}\Big|_{x=0}^{x=2\pi}}{2\pi} = \pi.$$

Hence, the Fourier series of x on $[0,2\pi]$ is:
$$f(x) = \boxed{\pi - \sum_{n=1}^{\infty}\frac{2}{n}\sin(nx)}.$$

2. (PDEs: Wave Equation). The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length L, the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x, t)$:
$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x, 0) = \sin\left(\dfrac{2\pi x}{L}\right), & \\ & \dfrac{\partial u}{\partial t}(x, 0) = \sin\left(\dfrac{5\pi x}{L}\right), & \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0, t) = u(L, t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$
where c is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*, $u(x, t) = v(x) \cdot w(t)$, and attempt to obtain a general solution that is *non-trivial*.
Hint: Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L), 1\}_{n \in \mathbb{Z}^+}$ forms an orthogonal basis (Question 1).

Solution:
With the method of separation, we insert the separations back to the system of equation to obtain:
$$v(x)w''(t) = c^2 v''(x)w(t).$$
Now, we apply the separation and set the common ratio to be λ:
$$\frac{v''(x)}{v(x)} = \frac{1}{c^2} \cdot \frac{w''(t)}{w(t)} = \lambda.$$
Reformatting the boundary condition gives use the following initial value problem:
$$\begin{cases} v''(x) - \lambda v(x) = 0, \\ v(0) = v(L) = 0. \end{cases}$$
As a second order linear ordinary differential equation, we discuss all following cases:

- If $\lambda = 0$, then $v(x) = a + Bx$ and by the initial condition, $A = B = 0$, which gives the trivial solution, *i.e.*, $v(x) = 0$;
- If $\lambda = \mu^2 > 0$, then we have $v(x) = Ae^{-\mu x} + Be^{\mu x}$ and again giving that $A = B = 0$, or the trivial solution;
- Eventually, if $\lambda = -\mu^2 < 0$, then we have the solution as:
$$v(x) = A\sin(\mu x) + B\cos(\mu x),$$
and the initial conditions gives us that:
$$\begin{cases} v(0) = B = 0, \\ v(L) = A\sin(\mu L) + B\cos(\mu L) = 0, \end{cases}$$
where A is arbitrary, $B = 0$, and $\mu L = m\pi$ positive integer m.

Overall, the only non-trivial solution would be:
$$v_m(x) = A\sin(\mu_m x), \text{ where } \mu_m = \frac{m\pi}{L}.$$
Eventually, by inserting back $\lambda = -\mu_m^2$, we have $\lambda = -m^2\pi^2/L^2$, giving the solution to $w_m(t)$, another second order linear ordinary differential equation, as:
$$w_m(t) = C\cos(\mu_m ct) + D\sin(\mu_m ct), \text{ with } C, D \in \mathbb{R}.$$
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By the *principle of superposition*, we can have our solution in the form:

$$u(x,t) = \sum_{m=1}^{\infty} \left[a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)\right] \sin(\mu_m x),$$

where our coefficients a_m and b_m have to be chosen to satisfy the initial conditions for $x \in [0, L]$:

$$u(x,0) = \sum_{m=1}^{\infty} a_m \sin(\mu_m x) = \sin\left(\frac{2\pi x}{L}\right),$$
$$\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} c\mu_m b_m \sin(\mu_m x) = \sin\left(\frac{5\pi x}{L}\right).$$

Since we are hinted that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis, we now know that except for the following:

$$a_2 = 1 \text{ and } c\mu_5 b_5 = 1,$$

all the other coefficients are zero, so we have:

$$u(x,t) = \boxed{\cos\left(\frac{2\pi ct}{L}\right)\sin\left(\frac{2\pi x}{L}\right) + \frac{L}{5\pi c}\sin\left(\frac{5\pi ct}{L}\right)\sin\left(\frac{5\pi x}{L}\right)}.$$

3. (*Putnam 2023*: A Linear System). Determine the smallest positive real number r such that there exists differentiable functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ satisfying:
   - $f(0)>0$,
   - $g(0)=0$,
   - $|f'(x)|\le|g(x)|$ for all x,
   - $|g'(x)|\le|f(x)|$ for all x, and
   - $f(r)=0$.

   You may give an answer *without* a rigorous proof, as the proof is out of scope of the course.
   Hint: Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating f and g.

Solution:
Here, we first provide a "simplified" case, *i.e.*, we are constructing a dynamical system in which we pick equality for the inequality, that is:

$$\begin{cases}|f'(x)|=|g(x)|, \text{ and}\\ |g'(x)|=|f(x)|.\end{cases}$$

Without loss of generality, we may assume that f and g are non-negative before r, so the system becomes:

$$\begin{cases}f'=-g\\ g'=f\end{cases},$$

or equivalently, $\mathbf{y}=\begin{pmatrix}f\\ g\end{pmatrix}$ that $\mathbf{y}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathbf{y}$. Clearly, we observe the eigenvalues are $\pm\mathrm{i}$ as the polynomial is $\lambda^2+1=0$. Moreover, the eigenvectors for $\lambda_1=\mathrm{i}$ is when $\begin{pmatrix}-\mathrm{i}&-1\\1&-\mathrm{i}\end{pmatrix}\boldsymbol{\xi}=\mathbf{0}$, in which we have $\boldsymbol{\xi}=y\begin{pmatrix}\mathrm{i}\\1\end{pmatrix}$, and that solution is:

$$\mathbf{y}=\begin{pmatrix}\mathrm{i}\\1\end{pmatrix}e^{\mathrm{i}x}=\begin{pmatrix}\mathrm{i}\\1\end{pmatrix}(\cos x+\mathrm{i}\sin x)=\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix}+\mathrm{i}\begin{pmatrix}\cos x\\ \sin x,\end{pmatrix}$$

and by conjugation, the solution should be:

$$\begin{pmatrix}f\\ g\end{pmatrix}=C_1\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix}+C_2\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}.$$

Note that with the given initial condition that $g(0)=0$, this enforces $C_1=0$, thus $f(x)=C\cos x$ and $g(x)=C\sin x$, and we know that $f(r)$ is zero first at $r=\boxed{\pi/2}$.

The above version has some reasoning, but is not a rigorous proof at all, since this does not consider if r could be smaller than $\pi/2$. For students with interests, we provide the complete proof from the Putnam competition from Victor Lie, as follows.

Proof. Without loss of generality, we assume $f(x)>0$ for all $x\in[0,r)$ as it is the first positive zero. By the fundamental theorem of calculus, we have:

$$|f'(x)|\le|g(x)|\le\left|\int_0^x g(s)ds\right|\le\int_0^x|g(s)|ds\le\int_0^t|f(s)|ds.$$

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Now, as we denote $F(x) = \int_0^x f(s)ds$, we have:

$$f'(x) + F(x) \geq 0 \text{ for } x \in [0, r].$$

For the sake of contradiction, we suppose $r < \pi/2$, then we have:

$$f'(x) \cos x + F(x) \cos x \geq 0 \text{ for } x \in [0, r].$$

Notice that the left hand side is the derivative of $f(x) \cos x + F(x) \sin x$, so an integration on $[y, r]$ gives:

$$F(r) \sin r \geq f(y) \cos y + F(y) \sin(y).$$

With some rearranging, we can have:

$$F(r) \sin r \sec^2 y \geq f(y) \sec y + F(y) \sin y \sec^2 y$$

Again, we integrate both sides with respect to y on $[0, r]$, which gives:

$$F(r) \sin^2 r \geq F(r),$$

and this is impossible, so we have a contradiction.
Hence we must have $r \geq \pi/2$, and since we have noted the solution $f(x) = C \cos x$ and $g(x) = C \sin x$, we have proven that $r = \pi/2$ is the smallest case. □

4. (Nilpotent Operator). Let M be a square matrix, M is *nilpotent* if $M^k = 0$ for some positive integer k. Similar to how we defined the exponential function analytically, the exponential function is also defined for matrices, let A be a square matrix, we define:

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i.$$

(a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

(b) Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
Hint: Use the differences of squares for matrices.

(c) If all the entries in N are rational, show that $\exp(N)$ has rational entries.

Solution:

(a) *proof of N is nilpotent.* Here, we want to do the matrix multiplication:

$$N^2 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N^3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we have shown that $N^3 = 0$, or the zero matrix, hence N is nilpotent. □

Then, we want to calculate the matrix exponential, that is:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}.$$

(b) *Proof.* Here, we recall the differences of squares still works when commutativity for multiplications fails, hence the we can still use it for matrix multiplication, namely, for all $m \in \mathbb{Z}^+$:

$$(\mathrm{Id}_n + N) \cdot (\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) = \mathrm{Id}_n - N^{2^{m+1}}$$

Since N is *nilpotent*, this implies that we have some k such that $N^\ell = 0$ for all $\ell \geq k$. Meanwhile, note that $2^\ell \geq \ell$ for all positive integer ℓ. (This can be proven by induction.) Therefore, we select $m + 1 \geq k$ so that $N^{2m+1} = 0$, and we have:

$$(\mathrm{Id}_n + N) \cdot \left[(\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) \right] = \mathrm{Id}_n,$$

thus $\mathrm{Id}_n + N$ is invertible. □

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(c) *Proof.* By the definition that N is nilpotent, we know that $N^m = 0$ for some finite positive integer m, hence, we can make the (countable) infinite sum into a finite sum:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \sum_{k=0}^{m} \frac{1}{k!} N^k,$$

thus all the entries are sum and non-zero divisions of rational numbers, while rational numbers are closed under addition and non-zero divisions, hence, all entries of $\exp(N)$ is rational. □

Note that the elements of all n-by-n matrices can be considered as a *ring*, while *nilpotent* can be defined more generally for *rings*. We invite capable readers to investigate more properties of *nilpotent* elements of *rings* in the discipline of *Modern Algebra*.

5. (Rotational Matrix). Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle θ (counter-clockwise), then:
$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

   (a) Show that $M^\intercal = M^{-1}$.

   (b) Let $\theta = 2\pi/k$ be fixed, where k is an integer. Find the least positive integer n such that $M^n = \mathrm{Id}_2$. Here, n is called the *order* of M.
   Hint: Consider the rotational matrix geometrically, rather than arithmetically.

   (c) Let $\theta = \pi/2$, calculate the matrix exponential $\exp(M)$.
   Hint: Consider the *order* of M and the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$ (cf. §1.2).

Solution:

(a) *Proof.* Here, we recall the method of inverting a matrix:
$$M^{-1} = \frac{1}{\det M}\begin{pmatrix} \cos\theta & -(-\sin\theta) \\ -\sin\theta & \cos\theta \end{pmatrix} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = M^\intercal.$$
□

(b) Look, we want to analyze this geometrically, if $\theta = 2\pi/k$, then that implies that M is a counter-clockwise rotation of $2\pi/k$, and since a full revolution is 2π, this implies a rotation of k times will make restore to the original vector, *i.e.*, $M^k = \mathrm{Id}_2$. Moreover, for any positive integer less than k, we cannot rotate back to 2π, which implies that the order of M is $\boxed{k}$.

(c) Here, we construct the matrix exponential, note that the order of M is 4, we have:
$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$
Here, we want to consider each entry respectively, since each entry is finite and since M has order 4, the absolute value of the sum of the entries must be finite, so each entry converges *absolutely*, hence we are free to change the order of the sum, so we have:
$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{(4k+1)!} M + \sum_{k=0}^{\infty} \frac{1}{(4k+2)!} M^2 + \sum_{k=0}^{\infty} \frac{1}{(4k+3)!} M^3 + \sum_{k=0}^{\infty} \frac{1}{(4k)!} \mathrm{Id}.$$
For the 4 sums of factorials, we note that the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$ at 0 evaluated at $x = 1$ are, respectively:
$$\begin{array}{rcccccccl} e^1 & = & +\frac{1}{0!} & +\frac{1}{1!} & +\frac{1}{2!} & +\frac{1}{3!} & +\frac{1}{4!} & +\frac{1}{5!} & +\cdots \\ e^{-1} & = & +\frac{1}{0!} & -\frac{1}{1!} & +\frac{1}{2!} & -\frac{1}{3!} & +\frac{1}{4!} & -\frac{1}{5!} & +\cdots \\ \sin 1 & = & & +\frac{1}{1!} & & -\frac{1}{3!} & & +\frac{1}{5!} & -\cdots \\ \cos 1 & = & +\frac{1}{0!} & & -\frac{1}{2!} & & +\frac{1}{4!} & & -\cdots \end{array}$$
Since the first series converges, we know that the later three series converges *absolutely*, so we are free to move around terms.

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From the expressions, by columns, we can observe that:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} = \frac{e^1 - e^{-1}}{4} + \frac{\sin 1}{2}, \qquad \sum_{k=0}^{\infty} \frac{1}{(4k+2)!} = \frac{e^1 + e^{-1}}{4} - \frac{\cos 1}{2},$$
$$\sum_{k=0}^{\infty} \frac{1}{(4k+3)!} = \frac{e^1 - e^{-1}}{4} - \frac{\sin 1}{2}, \qquad \sum_{k=0}^{\infty} \frac{1}{(4k)!} = \frac{e^1 + e^{-1}}{4} + \frac{\sin 1}{2}.$$

Now, we shall also evaluate the matrices generated by M, that is:

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$M^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, considering the four entries $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have:

$$a = -\frac{e+1/e}{4} + \frac{\cos 1}{2} + \frac{e+1/e}{4} + \frac{\sin 1}{2} = \frac{\cos 1 + \sin 1}{2},$$
$$b = -\frac{e-1/e}{4} - \frac{\sin 1}{2} + \frac{e-1/e}{4} - \frac{\sin 1}{2} = -2\sin 1,$$
$$c = \frac{e-1/e}{4} + \frac{\sin 1}{2} - \frac{e-1/e}{4} + \frac{\sin 1}{2} = 2\sin 1,$$
$$d = -\frac{e+1/e}{4} + \frac{\cos 1}{2} + \frac{e+1/e}{4} + \frac{\sin 1}{2} = \frac{\cos 1 + \sin 1}{2}.$$

Therefore, the matrix exponential is:

$$\exp(M) = \boxed{\begin{pmatrix} \frac{\cos 1 + \sin 1}{2} & -2\sin 1 \\ 2\sin 1 & \frac{\cos 1 + \sin 1}{2} \end{pmatrix}}.$$

In particular, mathematicians has considered the *rotation* and *flipping* of regular polygons as the *dihedral groups*, where symmetries and combinatorics play an important role. Please think of ways you may "manipulate" a polygon such that the polygon looks the same.