

PILOT Final Exam Review

Differential Equations

Johns Hopkins University

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As you prepare for Final exam, please consider the following resources:

- PILOT webpage for ODEs:
<https://jhu-ode-pilot.github.io/SP26/>
 - Find all the weekly sets and review problem sets.
 - Please also reuse previous review problem sets since the final is cumulative.
 - Consult the archives page for PILOT sets from the semester.
- Review the *homework/worksheets/quizzes* provided by the instructor.
- Join the PILOT Final Review Session. (You are here.)

Part 1:

Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

1 Preliminaries

2 First Order ODEs

3 Second Order ODEs

4 Higher Order ODEs

5 System of First Order Linear ODEs

6 Non-linear Systems

Preliminaries

- Classifications of Differential Equations
 - ODEs vs PDEs
- Modeling Using ODEs
 - Half Life Problem

When having various differential equations, we can classify them by their properties.

ODEs vs PDEs

Ordinary Differential Equations (ODEs) involves ordinary derivatives ($\frac{dy}{dt}$), while Partial Differential Equations (PDEs) involves partial derivatives ($\frac{\partial y}{\partial t}$).

This course focuses on ODEs, and it can also be classified in various different ways:

- **Single equation** involves one unknown and one equation, while **system of equations** involves multiple unknowns and multiple equations.
- The **order** of the differential equation is the order of the highest derivatives term.
- **Linear** differential equations has only linear dependent on the function, while **non-linear** differential equations has non-linear dependent on the function.

ODEs can be used for modeling. During modeling, it often follows the following steps:

- 1 Construction of the Models,
- 2 Analysis of the Models,
- 3 Comparison of the Models with Reality.

An example of modeling is the **half-life problem**.

Half Life Problem

The physics model for half life indicates the relationship between half life (τ) of a substance of amount $N(t)$ with initial amount N_0 at a time t is:

$$N(t) = N_0 \left(\frac{1}{2} \right)^{\frac{t}{\tau}},$$

where the rate of decay (λ) and half life (τ) are related by:

$$\tau \times \lambda = \log 2.$$

First Order ODEs

- Methods of Solving ODEs
 - Separable ODEs
 - Integrating Factor
- Existence and Uniqueness Theorems
- Autonomous ODEs
 - Rational Root Test
- Logistic Population Growth
 - Partial Fractions
- Exactness Problem
 - Integrating Factor for Non-Exact Case

Here, we will introduce various ways of solving ODEs:

Separable ODEs

For ODEs in form $M(t) + N(y) \frac{dy}{dt} = 0$, it can be separated by:

$$M(t)dt + N(y)dy = 0.$$

When the ODE is not separable, we may consider using the **integrating factor**.

Integrating Factor

For ODEs in form $\frac{dy}{dt} + a(t)y = b(t)$, the integrating factor is:

$$\mu(t) = \exp\left(\int a(t)dt\right).$$

The existence and uniqueness for Initial Value Problem (IVP) tells us information on if we can obtain a unique solution over some interval:

- For an IVP in simple form:

$$\begin{cases} \frac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$

If $a(t)$ and $b(t)$ are continuous on an interval (α, β) and $t_0 \in (\alpha, \beta)$. Then, there exists a uniqueness solution y for (α, β) to the IVP.

Picard's Theorem

- For an IVP in general form:

$$\begin{cases} \frac{dy}{dt} = f(t, y), \\ y(t_0) = y_0. \end{cases}$$

For $t_0 \in I = (a, b)$, $y_0 \in J = (c, d)$, if $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous on interval $I \times J$. Then, there exists a unique solution on a smaller interval $I' \times J' \subset I \times J$, in which $(t_0, y_0) \in I' \times J'$.

Only Contrapositive is Guaranteed to be True

For both theorems, you can conclude that if *there does not exist a solution or the solution is not unique*, then *the conditions must not be satisfied*. You **cannot** conclude that if *the conditions are not satisfied*, then *there is no unique solution*.

Autonomous ODEs are in form of:

$$\frac{dy}{dt} = f(y).$$

The stability (stable/semi-stable/unstable) of equilibrium can be determined by phase lines, *i.e.*, the zeros of the function $f(t)$.

Rational Root Test

Let the polynomial with integer coefficients be defined as:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0,$$

then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p|a_0$ and $q|a_n$.

The logistic population growth model with population (y), growing rate (r), and carrying capacity (k) is given by:

$$\begin{cases} \frac{dy}{dt} = r \left(1 - \frac{y}{k}\right) y, \\ y(0) = y_0, \end{cases}$$

whose general solution is $y(t) = \frac{ky_0}{(k - y_0)e^{-rt} + y_0}$.

Partial Fractions

For a fraction in the form $\frac{C}{(x - a_1)^{n_1}(x - a_2)^{n_2} \cdots (x - a_m)^{n_m}}$, it can be decomposed in terms of:

$$\frac{C_{1,1}}{x - a_1} + \frac{C_{1,2}}{(x - a_1)^2} + \cdots + \frac{C_{1,n_1}}{(x - a_1)^{n_1}} + \cdots + \frac{C_{m,1}}{x - a_m} + \cdots + \frac{C_{m,n_m}}{(x - a_m)^{n_m}}.$$

The condition for a function in form $M(x, y) + N(x, y) \frac{dy}{dx} = 0$ to be exact is:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

For solving Exact ODEs, either finding $\int M(x, y)dx + h(y)$ or $\int N(x, y)dy + h(x)$ and taking partials again to fit gives the solution $\Psi(x, y) = C$.

Integrating Factor for Non-Exact Case

$$\mu(t) = \exp\left(\int \frac{M_y - N_x}{N} dx\right) \quad \text{or} \quad \mu(t) = \exp\left(\int \frac{N_x - M_y}{M} dy\right).$$

Second Order ODEs

- Linear Homogeneous Cases
 - Complex Characteristic Roots
 - Repeated Characteristic Roots
- Linear Independence
 - Definition of Linearly Independence
 - Superposition Principle
- Reduction of Order
 - Product Rule and Chain Rule
- Non-homogeneous Cases
 - Variation of Parameters
 - Undetermined Coefficients

Consider the linear homogeneous ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is $r^2 + pr + q = 0$, with real, distinct solutions r_1 and r_2 , the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t} \cos(\beta t) + c_2 e^{\lambda t} \sin(\beta t).$$

Repeated Characteristic Roots

If the solutions are repeated, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

To form a fundamental set of solutions, the solutions need to be linearly independent, in which the Wronskian (W) must be non-zero, meaning that:

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \dots, f_n, \dots\}$ is linearly independent when for $\lambda_1, \lambda_2, \dots, \lambda_n, \dots \in \mathbb{F}$ (typically \mathbb{C}):

$$\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_n f_n + \dots = 0 \iff \lambda_1 = \lambda_2 = \dots = \lambda_n = \dots = 0.$$

Superposition Principle

If $y_1(t)$ and $y_2(t)$ are solutions to $L[y] = 0$, then the solution $c_1 y_1(t) + c_2 y_2(t)$ are also solutions for all constants $c_1, c_2 \in \mathbb{R}$.

For non-linear second order homogeneous ODEs, when one solution $y_1(t)$ is given, the other solution is in form:

$$y_2(t) = u(t) \cdot y_1(t).$$

Product Rule and Chain Rule

- **Product Rule:** $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x)g(x) + f(x)\frac{dg}{dx}(x).$
- **Chain Rule:** $\frac{d}{dx}[f(g(x))] = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).$

Procedure of Reduction of Order

As long as $y_1(t)$ is a solution, you will be able to reduce the differential equation with respect to y_2 into a differential equation involving only $u''(t)$ and $u'(t)$ terms to solve for $\omega(t) = u'(t)$.

Let the differential equation be:

$$Ay''(t) + By'(t) + Cy(t) = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$ and $y_2(t)$ be the two homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

Variation of Parameters

The particular solution of the differential equation can be written as the integrals of respective parts.

$$y_p = y_1(t) \int \frac{-y_2(t) \cdot g(t)}{W} dt + y_2(t) \int \frac{y_1(t) \cdot g(t)}{W} dt.$$

Another approach is less calculation intensive, but requires the function $g(t)$ to be constrained in certain forms.

Undetermined Coefficients

A guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$	Guess
Polynomials: $\sum_{i=0}^d a_i t^i$	$\sum_{i=0}^d C_i t^i$
Trig.: $\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \cos(ax)$
Exp.: e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

Higher Order ODEs

- Existence and Uniqueness Theorem
- Homogeneous Cases
 - Complex Characteristic Roots
 - Repeated Characteristic Roots
- Linear Independence
 - Definition of Linearly Independence
- Abel's Formula
- Non-Homogeneous Cases
 - Undetermined Coefficients

For higher order IVP in form:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

If $P_0(t), P_1(t), \cdots, P_{n-1}(t)$, and $g(t)$ are continuous on an interval I containing t_0 . Then there exists a unique solution for $y(t)$ on I .

Only Contrapositive is Guaranteed to be True

Again, for this theorem, you can conclude that if *there does not exist a solution or the solution is not unique, then the conditions must not be satisfied*. You **cannot** conclude that if *the conditions are not satisfied, then there is no unique solution*.

The higher order homogeneous ODEs are in form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

By computing the characteristic equation

$r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$, with solutions r_1, r_2, \dots, r_n ,
the general solution is $y(t) = c_1e^{r_1t} + c_2e^{r_2t} + \cdots + c_ne^{r_nt}$.

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1e^{\lambda t} \cos(\beta t) + c_2e^{\lambda t} \sin(\beta t) + \text{rest of the solutions.}$$

Repeated Characteristic Roots

If the solutions are repeated with multiplicity m , the solution is:

$$y(t) = c_1e^{rt} + c_2te^{rt} + \cdots + c_mt^{m-1}e^{rt} + \text{rest of the solutions.}$$

To obtain the fundamental set of solutions, the Wronskian (W) must be non-zero, where Wronskian is:

$$W[y_1, y_2, \dots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \dots, f_n, \dots\}$ is linearly independent when for $\lambda_1, \lambda_2, \dots, \lambda_n, \dots \in \mathbb{F}$ (typically \mathbb{C}):

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n + \cdots = 0 \iff \lambda_1 = \lambda_2 = \cdots = \lambda_n = \cdots = 0.$$

For higher order ODEs in the form of:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

Its Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = Ce^{\int -P_{n-1}(t)dt},$$

where C is independent of t but depend on y_1, y_2, \cdots, y_n .

When $g(t) \neq 0$, we can also guess the particular solution.

Undetermined Coefficients

Same as in degree 2, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$	Guess
Polynomials: $\sum_{i=0}^d a_i t^i$	$\sum_{i=0}^d C_i t^i$
Trig.: $\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \cos(ax)$
Exp.: e^{at}	Ce^{at}

Again, the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

System of First Order Linear ODEs

- Solving for Eigenvalues and Eigenvectors
- Linear Independence
 - Abel's Formula
- Phase Portraits
 - Node Graph
- Repeated Eigenvalues
 - Algebraic Multiplicity and Geometric Multiplicity
- Phase Portraits
 - Node Graph
 - Spiral/Center Graph
 - Repeated Eigenvalue Graph

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - rI) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - rI) \cdot \boldsymbol{\zeta} = \mathbf{0}.$$

Suppose that the eigenvalues are distinct and the eigenvectors are linearly independent, the solution to the ODE is:

$$\mathbf{x} = c_1 \boldsymbol{\zeta}^{(1)} e^{r_1 t} + c_2 \boldsymbol{\zeta}^{(2)} e^{r_2 t} + \cdots + c_n \boldsymbol{\zeta}^{(n)} e^{r_n t}.$$

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det(\Psi(t)).$$

The system is linearly independent if the Wronskian is non-zero.

Abel's Formula

For the linear system in form:

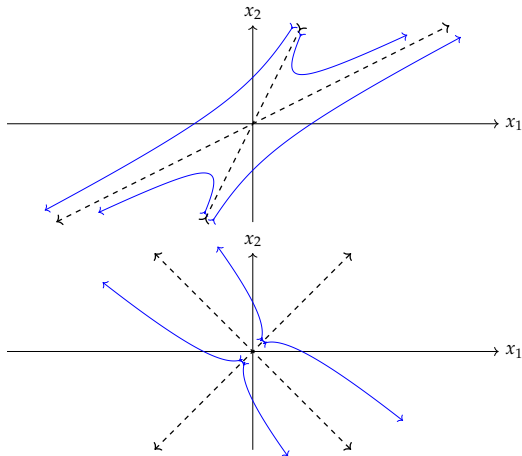
$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A , which is the sum of the diagonals, that is:

$$W = Ce^{\int \text{trace } A dt} = Ce^{\int (A_{1,1} + A_{2,2} + \dots + A_{n,n}) dt}.$$

In particular, we can sketch the linear system of \mathbb{R}^2 in terms of phase portraits given the eigenvalues and eigenvectors.

- For a node graph, we have it as (directions might vary):



For repeated eigenvalue r with only one (linearly independent) eigenvector, if a given a solution is $\mathbf{x}^{(1)} = \boldsymbol{\zeta}e^{rt}$, the other solution would be:

$$\mathbf{x}^{(2)} = \boldsymbol{\zeta}te^{rt} + \boldsymbol{\eta}e^{rt},$$

where $(A - Ir) \cdot \boldsymbol{\eta} = \boldsymbol{\zeta}$, and $\mathbf{x}^{(2)}$ is called the *generalized eigenvector*.

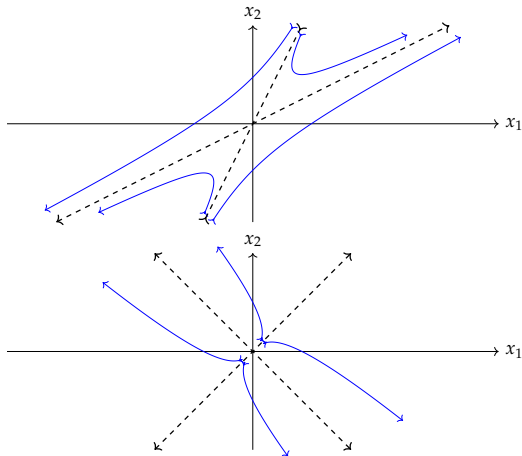
Algebraic Multiplicity and Geometric Multiplicity

The algebraic multiplicity refers to the multiplicity of root in the characteristic polynomial, and the geometric multiplicity refers to the dimension of the eigenspace associated with the eigenvalue.

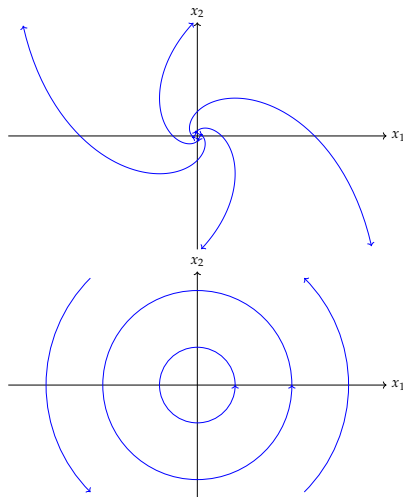
- The algebraic multiplicity will be no less than the geometric multiplicity for each eigenvalue.
- We need the generalized eigenvector when the algebraic multiplicity is larger than the geometric multiplicity.

In particular, we can sketch the linear system of \mathbb{R}^2 in terms of phase portraits given the eigenvalues and eigenvectors.

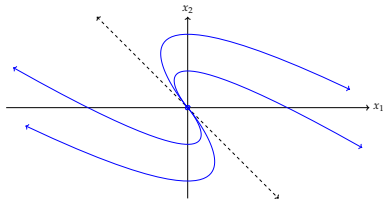
- For a node graph, we have it as (directions might vary):



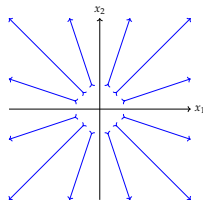
- For a spiral/center graph, we have it as (directions might vary):



- For repeated eigenvalues with less geometric multiplicity, the solution is (directions might vary):



- If the geometric multiplicity is the same, the graph is simply a radial shape (directions might vary):



Non-linear Systems

- Linear Approximation
 - Autonomous Systems
- Stability
- Limit Cycles

For non-linear system $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, if $F, G \in C^2$ and the system is locally linear, the approximation at critical point (x_0, y_0) is:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \mathbf{J}(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where Jacobian is:

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}.$$

Autonomous Systems

When $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} F(y) \\ G(x) \end{pmatrix}$, it can be solved implicitly for:

$$\frac{dy}{dx} = \frac{G(x)}{F(y)}.$$

For linearized system with eigenvalues r_1, r_2 , the stability can be concluded as follows:

Eigenvalues	Linear System		Nonlinear System	
	Type	Stability	Type	Stability
Eigenvalues are λ_1 and λ_2				
$0 < \lambda_1 < \lambda_2$	Node	Unstable	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable	Node or Spiral Point	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable	Node or Spiral Points	Asymptotically Stable
Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$				
$\alpha > 0$	Spiral Point	Unstable	Spiral Point	Unstable
$\alpha = 0$	Center	Stable	Center or Spiral Point	Indeterminate
$\alpha < 0$	Spiral Point	Asymptotically Stable	Spiral Point	Asymptotically Stable

Part 2: Open Poll

We will work out some sample questions.

- Please let us know if you have questions from the Review Problem Sets or Weekly Problem Sets.
- Let us know if you want to go through any concepts that you are not sure with.

Thank you for being with **Differential Equations PILOT** this semester.

Good luck on your finals, and best wished for your future pursuits!

While you prepare for the final:

Collaborate with your friends and classmates to study together!

Get physically and mentally prepared. Get enough sleep and some food before the exam!

Believe in yourself. You are amazing and you can make it this!