Final Review Problem Set: Solutions

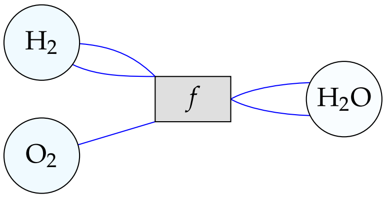
Differential Equations

Spring 2026

1. Consider the following chemical equation of hydrogen gas combustion in oxygen gas:

$$2H_2 + O_2 \xrightarrow{f} 2H_2O.$$

We may represent it from a graphical representation.



Assume that the reaction rate is constant $\kappa := \mathrm{rate}(f)$. Construct the nonlinear system of the concentration of H_2 and O_2, sketch a few trajectories for different initial conditions for different starting concentrations.

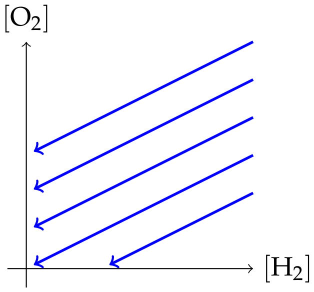
Solution:

For the model, we recall that the reaction rate is proportional to the concentration of the current amount of reactants. Hence, we obtain the system as:

$$\boxed{\begin{cases} \dfrac{d[H_2]}{dt} = -2\kappa[H_2][O_2], \\ \dfrac{d[O_2]}{dt} = -\kappa[H_2][O_2]. \end{cases}}$$

This is system is of course not linear, but we may notice that the rates are exactly the same, except for the linear factor $1/2$, so that means that the concentration of H_2 will be consuming twice as fast as the concentration of O_2 being consumed.

If we are to plot this, we should have the line with a slope of $1/2$, namely:



If you are interested in graphical (or more precisely *categorical*) representation of chemical equations, learn more about category theory.

2. Carbon-14, a radioactive isotope of carbon, is an effective tool in dating the age of organic compounds, as it decays with a relatively long period. Let $Q(t)$ denote the amount of carbon-14 at time t, we suppose that the decay of $Q(t)$ satisfies the following differential equation:

$$\frac{dQ}{dt} = -\lambda Q \text{ where } \lambda \text{ is the rate of decay constant.}$$

   (a) Let the half-life of carbon-14 be τ, find the rate of decay, λ.

   (b) Suppose that a piece of remain is discovered to have 10% of the original amount of carbon-14, find the age of the remain in terms of τ.

Solutions:

(a) Note that the differential equation is separable, hence:

$$\frac{dQ}{Q} = -\lambda dt,$$

$$\int \frac{dQ}{Q} = -\int \lambda dt,$$

$$\log |Q| = -\lambda t + C,$$

$$Q = \tilde{C} e^{-\lambda t}.$$

Here, we assume $Q = Q_0$ at $t = t_0$, then we have $Q = Q_0/2$ when $t = t_0 + \tau$, so:

$$\frac{1}{2} = e^{-\lambda \tau},$$

which deduces to:

$$\lambda = -\frac{1}{\tau} \log\left(\frac{1}{2}\right) = \boxed{\frac{\log 2}{\tau}}.$$

(b) If there are only 10% of remain, we suppose that we have $Q = Q_0$ at $t = 0$, and have $Q = Q_0/10$ at $t = t_0$, hence giving that:

$$\frac{Q_0}{10} = Q_0 \exp\left(-\lambda t_0\right) = Q_0 \exp\left(-\frac{\log(2) t_0}{\tau}\right).$$

Thus, we obtain that:

$$\frac{1}{10} = \exp\left(-\frac{\log(2) t_0}{\tau}\right),$$

and by solving for t_0, we obtain:

$$t_0 = -\frac{\tau}{\log 2} \log\left(\frac{1}{10}\right) = \boxed{\frac{\log 10}{\log 2}\tau}.$$

3. Find the solution to the following ODE:

$$\frac{y'}{x} + e^{x^2} y = e^{x^2}.$$

Solutions:

Here, we notice that we can transform the equation a little bit to use integrating factor:

$$y' + xe^{x^2} = xe^{x^2}.$$

Hence, the integrating factor is:

$$\mu(x) = \exp\left(\int xe^{x^2} dx\right) = \exp\left(\frac{1}{2}e^{x^2} + C\right) = \exp\left(\frac{1}{2}e^{x^2}\right).$$

Therefore, we multiple the integrating factor to have:

$$y' e^{e^{x^2}/2} + xe^{x^2} e^{e^{x^2}/2} y = xe^{e^{x^2}/2} e^{x^2}.$$

Thus, we have:

$$ye^{e^{x^2}/2} = \int xe^{e^{x^2}/2} e^{x^2} dx = e^{e^{x^2}/2} + C,$$

notice that the right hand side integration is exactly the integrating factor and the original part. Hence, we have the solution:

$$y(x) = \boxed{1 + Ce^{-e^{x^2}/2}}.$$

4. Consider the Hamiltonian potential as:

$$H(x,y) = x^2y + \frac{y^3}{3} + \sin x.$$

Construct the nonlinear system as:

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}, \\ y' = -\dfrac{\partial H}{\partial x}. \end{cases}$$

Find the trajectories in finding a relationship between x and y (as level curves).

Solutions:

Here, we first write the nonlinear system through the current scheme:

$$\begin{cases} x' = x^2 + y^2, \\ y' = -(2xy + \cos x). \end{cases}$$

Here, we write the differential form as:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{2xy + \cos x}{x^2 + y^2},$$

which can be written as:

$$\underbrace{(2xy + \cos x)}_{M(x,y)}\, dx + \underbrace{(x^2 + y^2)}_{N(x,y)}\, dy = 0.$$

Here, we notice that this ODE is not separable, but it is exact, *i.e.*:

$$\partial_y M = 2x \qquad \text{and} \qquad \partial_x N = 2x.$$

Hence, the exact trajectory is:

$$\phi(x,y) = \int (2xy + \cos x) dx = x^2y + \sin x + h(y).$$

Hence, when we take the partial derivative as:

$$\partial_y \phi(x,y) = x^2 + h'(y).$$

Hence, we have $h'(y) = y^2$, so $h(y) = \frac{1}{3}y^3$ and the trajectory is:

$$\boxed{x^2y + \sin x + \frac{y^3}{3} = C},$$

as level curves for $C \in \mathbb{R}$.

5. Find the solution to the following higher order ODE:

$$y''' - 6y'' + 12y' - 9y = 0.$$

Solutions:

Here, we first write the characteristic equation:

$$0 = r^3 - 6r^2 + 12r - 9 = r^3 - 6r^2 + 12r - 8 - 1 = (r-2)^3 - 1.$$

Hence, for $(r-2)^3 = 1$, we can have the roots as:

$$r = 2 + 1, \qquad 2 + e^{2\pi \mathrm{i}/3}, \qquad \text{and} \qquad 2 + e^{4\pi \mathrm{i}/3},$$

which are as:

$$r = 3, \frac{3}{2} + \mathrm{i}\frac{\sqrt{3}}{2}, \frac{3}{2} - \mathrm{i}\frac{\sqrt{3}}{2}.$$

Thus, the solutions are:

$$y(t) = \boxed{C_1 e^{3t} + C_2 e^{3t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + C_3 e^{3t/2} \sin\left(\frac{\sqrt{3}}{2} t\right)}.$$

6. Solve the following nonhomogeneous ODE:
$$(x^2+1)y''-2xy'+2y=x.$$
Hint: Observe a homogeneous solution, then *reduction of order*, eventually *variation of parameters*.

Solution:

First, we notice that $y=x$ is a solution and then, we can find the other solutions via reduction of order. Let $y_2(x)=u(x)y_1(x)$. Then
$$y_2'=u'x+u, \qquad y_2''=u''x+2u'.$$
Substitute it into the homogeneous equation gives:
$$\begin{aligned}0&=(x^2+1)(u''x+2v')-2x(u'x+v)+2(ux)\\&=x(x^2+1)u''+\left(2(x^2+1)-2x^2\right)u'=x(x^2+1)u''+2u'.\end{aligned}$$
Let $\omega=u'$. Then $x(x^2+1)\omega'+2\omega=0$. Notice it is separable, we have:
$$\frac{d\omega}{\omega}=-\frac{2dx}{x(x^2+1)}=\left(\frac{1}{x(x^2+1)}=\frac{1}{x}-\frac{x}{x^2+1}\right)dx.$$
Solving this differential equation gives
$$\begin{aligned}\log|\omega|&=\int\left(-\frac{2}{x}+\frac{2x}{x^2+1}\right)dx=-2\ln|x|+\ln(x^2+1)+C,\\ \omega&=C\frac{x^2+1}{x^2}=C\left(1+\frac{1}{x^2}\right).\end{aligned}$$
Therefore, we obtain u by integration, which gives:
$$u=\int\omega dx=C_1\left(x-\frac{1}{x}\right)+C_2,$$
and hence $y_2(x)=x^2-1$.

Then, we find the particular solution via variation of parameters. In standard form, we have:
$$y''-\frac{2x}{x^2+1}y'+\frac{2}{x^2+1}y=\frac{x}{x^2+1}=:g(x).$$
The Wronskian is:
$$W=y_1y_2'-y_1'y_2=x(2x)-1\cdot(x^2-1)=x^2+1.$$
Thus, variation of parameters gives the solutions as:
$$y_p(x)=-y_1(x)\int\frac{x(x^2-1)}{(x^2+1)^2}dx+y_2(x)\int\frac{x^2}{(x^2+1)^2}.$$
Then, we tackle each integration specifically:
$$\begin{aligned}\int\frac{x(x^2-1)}{(x^2+1)^2}dx&=\int\frac{x}{x^2+1}-\frac{2x}{(x^2+1)^2}dx=\frac{1}{2}\log(x^2+1)+\frac{1}{x^2+1},\\ \int\frac{x^2}{(x^2+1)^2}&=\int\frac{1}{x^2+1}-\frac{1}{(x^2+1)^2}dx=\arctan x-\int\frac{1}{(x^2+1)^2}dx\\&=\arctan x-\frac{1}{2}\int\frac{\sec^2 u}{(\tan^2 u+1)}du=\arctan x-\frac{1}{2}\int\cos^2 u du\\&=\arctan x-\frac{\sin 2u}{4}-\frac{u}{2}=\frac{1}{2}\arctan x-\frac{2x/(x^2+1)}{4}=\frac{1}{2}\arctan x-\frac{x}{2(x^2+1)}.\end{aligned}$$
Therefore, the general solution to the ODE is:
$$y(x)=\boxed{C_1x+C_2(x^2-1)-\frac{x}{2}\log(x^2+1)+\frac{x^2-1}{2}\arctan x}.$$

7. For the following non-linear systems, find all equilibrium(s) and classify their stability locally if they are locally linear.

(a)
$$\begin{cases} \dfrac{dx}{dt} = x - y^2, \\ \dfrac{dy}{dt} = x + x^2 - 2y. \end{cases}$$

(b)
$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y^2, \\ \dfrac{dy}{dt} = x + 4y^2. \end{cases}$$

Solution:

(a) For the first case, we notice that the equilibrium points are if:

$$\begin{cases} x - y^2 = 0, \\ x + x^2 - 2y = 0. \end{cases}$$

Note that this will be two parabolas, and there are at most two intersections, and we observe the intersections $(0,0)$ and $(1,1)$. Also to note, the Jacobian matrix is:

$$\mathrm{J} = \begin{pmatrix} 1 & -2y \\ 1+2x & -2 \end{pmatrix}.$$

- For the $(0,0)$ case, we denoting $\mathbf{x} = (x,y)$, we verify the linear approximation as:

  $$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x},$$

  and we note that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, and by:

  $$\lambda_2 < 0 < \lambda_1,$$

  we know that we have a unstable saddle point at $(0,0)$.

- For the $(1,1)$ case, we have the linear approximation as:

  $$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix},$$

  and we note the eigenvalues are $\lambda = \frac{-1 \pm i\sqrt{15}}{2}$, which is complex with a negative real part, so we have a $\boxed{\text{asymptotically stable spiral point}}$.

(b) Here, we note that the equilibrium(s) is achieved if and only if $x' = y' = 0$, that is:

$$\begin{cases} 2x + 3y^2 = 0, \\ x + 4y^2 = 0. \end{cases}$$

In particular, we consider $z = y^2$, so we have a system of linear equations, which simplifies to $x = y = 0$, hence the only equilibrium is at $(x,y) = (0,0)$.

Then, we consider the system locally, denoting $\mathbf{x} = (x,y)$, that is:

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x},$$

and note that the determinant of the Jacobian matrix is zero, so it is not locally linear, so we cannot conclude any information from this.

8. Let the following systems of (x, y) be functions of variable t:

(a)
$$\begin{cases} x' = (1+x)\sin y, \\ y' = 1 - x - \cos y. \end{cases}$$

(b)
$$\begin{cases} x' = x - y, \\ y' = x - 2y + x^2. \end{cases}$$

Identify the corresponding linear system, then evaluate the stability for the equilibrium at $(0, 0)$ by showing it is locally linear.

Solution:

(a) We evaluate x and y both at 0 for the differential equation, and $x' = y' = 0$, so $(0, 0)$ is a equilibrium. Then, we can find the Jacobian Matrix:

$$\mathrm{J} = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix},$$

and this implies that the linear system is:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}.$$

As we evaluate J at $(0, 0)$ and take its determinant, we have:

$$\det\left(\mathrm{J}\big|_{(0,0)}\right) = \det\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 \neq 0.$$

Hence, the $(0, 0)$ is locally linear.

Here, we have the eigenvalues as $\lambda^2 + 1 = 0$, so they are purely imaginary, so we have an $\boxed{\text{indeterminate spiral or center point}}$.

(b) We evaluate x and y both at 0 for the differential equation, and $x' = y' = 0$, so $(0, 0)$ is a equilibrium. Then, we consider the Jacobian matrix as:

$$\mathrm{J} = \begin{pmatrix} 1 & -1 \\ 2x+1 & -2 \end{pmatrix}.$$

Now, we evaluate the matrix at $(0, 0)$ and take its determinant:

$$\det(\mathrm{J}\big|_{(0,0)}) = \det\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} = -1 \neq 0.$$

Hence, the system is locally linear, and the linear system locally at $(0, 0)$ should be:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

We find its eigenvalue as:

$$0 = \det\begin{pmatrix} 1-\lambda & -1 \\ 1 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda) + 1 = \lambda^2 + \lambda - 1.$$

By using the quadratic formula, we have the eigenvalues as $\lambda = \frac{-1\pm\sqrt{5}}{2}$.

Thus, we have $\lambda_1 < 0 < \lambda_2$, so we have a $\boxed{\text{unstable saddle point}}$.