



Midterm 2 Review Problem Set: Solutions

Differential Equations

Spring 2026

1. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

(a) $y'' + 2y' + y = e^{-t}$.

(b) $y'' + y = \tan t$.

Solution:

(a) First, we look for homogeneous solution, *i.e.*, $y'' + 2y' + y = 0$, whose characteristic equation is:

$$r^2 + 2r + 1 = (r + 1)^2 = 0,$$

with root(s) being -1 with multiplicity of 2, so the general solution to homogeneous case is:

$$y_g(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

Notice that the non-homogeneous part is e^{-t} , but we have e^{-t} and $t e^{-t}$ as general solutions already, so we have our guess of particular solution as:

$$y_p(t) = A t^2 e^{-t}.$$

By taking the derivatives, we have:

$$y_p'(t) = A(2t e^{-t} - t^2 e^{-t}) \quad \text{and} \quad y_p''(t) = A(2e^{-t} - 4t e^{-t} + t^2 e^{-t}).$$

We simply plug in the particular solution, so we have:

$$\begin{aligned} A(2e^{-t} - 4t e^{-t} + t^2 e^{-t}) + 2A(2t e^{-t} - t^2 e^{-t}) + A t^2 e^{-t} &= e^{-t} \\ 2A e^{-t} &= e^{-t} \\ A &= \frac{1}{2}. \end{aligned}$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}}.$$

(b) Here, we still look for homogeneous solutions, *i.e.*, $y'' + y = 0$, whose characteristic equation is:

$$r^2 + 1 = 0,$$

with roots $\pm i$. Since we are dealing with real valued functions, we have the general solution as:

$$y_g = C_1 \sin t + C_2 \cos t.$$

Note that $\tan t$ does not work with undetermined coefficients, we must use the variation of parameters, the Wronskian of our solution is:

$$W[\sin t, \cos t] = \det \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} = -\sin^2 t - \cos^2 t = -1.$$

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Now, we may use the formula, namely getting the particular solution as:

$$\begin{aligned}y_p &= \sin t \int \frac{-\cos t \cdot \tan t}{-1} dt + \cos t \int \frac{\sin t \cdot \tan t}{-1} dt \\&= \sin t \int \sin t dt - \cos t \int \frac{\sin^2 t}{\cos t} dt \\&= \sin t(-\cos t + C) - \cos t \int \frac{1 - \cos^2 t}{\cos t} dt \\&= -\sin t \cos t - \cos t \left(\int \sec t dt - \int \cos t dt \right) \\&= -\sin t \cos t - \cos t (\log |\sec t + \tan t| - \sin t + C) \\&= -\sin t \cos t + \sin t \cos t - \cos t \log |\sec t + \tan t| \\&= -\cos t \log |\sec t + \tan t|.\end{aligned}$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 \sin t + C_2 \cos t - \cos t \log |\sec t + \tan t|}.$$

2. Solve for the general solution to the following higher order ODE.

(a)
$$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

(b)
$$\frac{d^4y}{dx^4} + y = 0.$$

Hint: Consider the 8-th root of unity, i.e., ζ_8 , and verify which roots satisfies the polynomial.

Solution:

(a) Note that we obtain the characteristic equation as:

$$4r^4 - 24r^3 + 45r^2 - 29r + 6 = 0.$$

To obtain our roots, we use the **Rational Root Theorem**, so if the characteristic equation has any rational root, it must have been one (or more) of the following:

$$\pm 1, \pm 2, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}.$$

From plugging in the values, we notice that 2 and 3 are roots of the characteristic equation, by division, we have:

$$\frac{4r^4 - 24r^3 + 45r^2 - 29r + 6}{(r - 2)(r - 3)} = 4r^2 - 4r + 1 = (2r - 1)^2.$$

Now, we know that the roots are 2, 3, and 1/2 with multiplicity 2, thus the solution to the differential equation is:

$$y(x) = \boxed{C_1e^{2x} + C_2e^{3x} + C_3e^{x/2} + C_4xe^{x/2}}.$$

*Again, we invite readers to verify the **Rational Root Theorem**.*

(b) For this general solution, we trivially obtain that the characteristic polynomial is:

$$r^4 + 1 = 0.$$

Recall that the root of unity address for the case when $r^n = 1$, so we consider the 8th root of unity, in which $(\zeta_8)^8 = 1$. Now, recall **Euler's Identity** and **deMoivre's formula**, we note that only the odd powers of the 8th root of unity satisfies that $r^4 = -1$, namely, are:

$$\zeta_8 = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2},$$

$$\zeta_8^3 = \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2},$$

$$\zeta_8^5 = \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2},$$

$$\zeta_8^7 = \cos\left(\frac{7\pi}{4}\right) + i \sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}.$$

Also, we note that ζ_8 and ζ_8^7 are complex conjugates, whereas ζ_8^3 and ζ_8^5 are complex conjugates, so we can linearly combine them to obtain the set of linearly independent solutions, i.e.:

$$y(x) = \boxed{e^{(\sqrt{2}/2)x} \left[C_1 \cos\left(\frac{\sqrt{2}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{2}}{2}x\right) \right] + e^{-(\sqrt{2}/2)x} \left[C_3 \cos\left(\frac{\sqrt{2}}{2}x\right) + C_4 \sin\left(\frac{\sqrt{2}}{2}x\right) \right]}.$$

3. Given the following second order initial value problem:

$$\begin{cases} \frac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \quad \frac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, i.e., satisfying that $y(x) = y(2-x)$.

Hint: Consider the interval in which the solution is unique.

Solution:

Note that I deliberately messed up with all the messy functions. Not only haven't I found a solution to the system, Wolfram cannot have an elementary solution as well. Hence, we need to think, alternatively, on some theorems.

Proof. Here, we suppose that $y(x)$ is a solution, and we want to show that $y(2-x)$ is also a solution. First we note that we can think of taking the derivatives of $y(2-x)$, by the chain rule:

$$\begin{aligned} \frac{d}{dx}[y(2-x)] &= -y'(2-x), \\ \frac{d^2}{dx^2}[y(2-x)] &= y''(2-x). \end{aligned}$$

Now, if we plug in $y(2-x)$ into the system of equations, we have:

- First, for the differential equation, we have:

$$\begin{aligned} \frac{d^2}{dx^2}[y(2-x)] + \cos(1-x)y(2-x) &= y''(2-x) + \cos(1-x)y(2-x) \\ &= y''(2-x) + \cos(1-(2-x))y(2-x) \\ &= y''(x) + \cos(1-x)y(x) \\ &= x^2 - 2x + 1 = (x-1)^2 = (1-x)^2 \\ &= ((2-x)-1)^2 = (2-x)^2 - 2(2-x) + 1. \end{aligned}$$

- For the initial conditions, we trivially have that:

$$y(1) = y(2-1) \text{ and } y'(1) = y'(2-1).$$

Hence, we have shown that $y(2-x)$ is a solution if $y(x)$ is a solution.

Again, we observe the original initial value problem that:

$$\cos(1-x) \text{ and } x^2 - 2x + 1 \text{ are continuous on } \mathbb{R}.$$

Therefore, by the *existence and uniqueness theorem for second order linear case*, there could be only one solution, which forces that:

$$y(x) = y(2-x),$$

so the solution is symmetric about $x = 1$, as desired. □

4. Let a system of differential equations of $x_i(t)$ be as follows:

$$\begin{cases} x_1' = 3x_1 + 2x_2, & x_1(1) = 0, \\ x_2' = x_1 + 4x_2, & x_2(1) = 2. \end{cases}$$

(a) Solve for the solution to the initial value problem.

(b) Identify and describe the stability at equilibrium(s).

Solution:

(a) Here, we denote $\mathbf{x} = (x_1 \ x_2)^T$, so our system becomes:

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Here, the eigenvalues are solutions to:

$$\det \begin{pmatrix} 3 - \lambda & 2 \\ 1 & 4 - \lambda \end{pmatrix} = 0,$$

which simplifies to $\lambda^2 - 7\lambda + 10 = 0$, and further gives $\lambda_1 = 2$, $\lambda_2 = 5$. Then, we look for eigenvectors of the matrix:

- For $\lambda_1 = 2$, we have $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which gives that $\boldsymbol{\xi}_1 = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = 5$, we have $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$, which gives that $\boldsymbol{\xi}_2 = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Now, the general solution must be:

$$\mathbf{x} = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t},$$

and by plugging in the initial condition, we have:

$$\begin{cases} -2C_1 e^2 + C_2 e^5 = 0, \\ C_1 e^2 + C_2 e^5 = 2. \end{cases}$$

In which the solution is $C_1 = \frac{2}{3e^2}$ and $C_2 = \frac{4}{3e^5}$, so the solution is:

$$\begin{cases} x_1 = -\frac{4}{3}e^{2t-2} + \frac{4}{3}e^{5t-5}, \\ x_2 = \frac{2}{3}e^{2t-2} + \frac{4}{3}e^{5t-5}. \end{cases}$$

(b) Now, we consider the equilibrium at $\mathbf{x} = (0 \ 0)^T$, in which we note that both eigenvalues are positive, meaning that this is an unstable node.

5. Let systems of differential equations be defined as follows, find the general solutions to the equations:

$$(a) \quad \mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = (x_1, x_2).$$

$$(b) \quad \mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = (x_1, x_2).$$

$$(c) \quad \mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = (x_1, x_2, x_3).$$

Solution:

(a) Here, we notice that the linear system is diagonal, so we can simply solve for each entry as $x_1 = e^{3t}$ and $x_2 = e^{2t}$. Hence, the solution is:

$$\mathbf{x} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) We first solve for the eigenvalues and eigenvectors:

$$0 = \det \begin{pmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{pmatrix} = (3 - \lambda)(-1 - \lambda) + 8 = \lambda^2 - 2\lambda + 5.$$

Hence, the eigenvalues are $\lambda = 1 \pm 2i$. Since they are complex conjugates, we pick $\lambda_1 = 1 - 2i$, so $\zeta^{(1)}$ satisfies $\begin{pmatrix} 2 + 2i & -2 \\ 4 & -2 + 2i \end{pmatrix} \cdot \zeta^{(1)} = \mathbf{0}$, so we have $(2 + 2i)\zeta_1^{(1)} = 2\zeta_2^{(1)}$, so the eigenvector is $\zeta^{(1)} = (1, 1 + i)$. We can get our solution:

$$\begin{aligned} \mathbf{x} &= e^{(1-2i)t} \begin{pmatrix} 1 + i \\ 1 \end{pmatrix} = e^t (\cos(2t) - i \sin(2t)) \begin{pmatrix} 1 \\ 1 + i \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i e^t \begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}. \end{aligned}$$

Hence, the solution is:

$$\mathbf{x} = C_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + C_2 e^t \begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

(c) Again, we first find the eigenvalues of the equation, *i.e.*:

$$\det \begin{pmatrix} 1 - \lambda & 0 & 4 \\ 1 & 1 - \lambda & 3 \\ 0 & 4 & 1 - \lambda \end{pmatrix} = 0,$$

which is $(1 - \lambda)^3 + 16 - 12(1 - \lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = -(\lambda + 1)^2(\lambda - 5) = 0$.

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$.

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Now, we look for eigenvectors.

- For $\lambda_1 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \xi_1 = \mathbf{0}$, which is $x \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$.

- For $\lambda_2 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \eta = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$, which is $\eta = \begin{pmatrix} 4x \\ x+1 \\ -2x-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

- For $\lambda_3 = 5$, we have $\begin{pmatrix} -4 & 0 & 4 \\ 1 & -4 & 3 \\ 0 & 4 & -4 \end{pmatrix} \xi_3 = \mathbf{0}$, which is $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = C_1 e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + C_2 \left(t e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) + C_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

6. Solve the following initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Plot the phase portrait for the general problem, note the trajectory for the initial value.

Solution:

Here, we first find the eigenvalues for the matrix, that is:

$$\det \begin{pmatrix} 1 - \lambda & -4 \\ 4 & -7 - \lambda \end{pmatrix} = 0.$$

Therefore, the polynomial is $(1 - \lambda)(-7 - \lambda) + 16 = (\lambda + 3)^2 = 0$, hence the eigenvalues is $\lambda_1 = \lambda_2 = -3$. Then, we look for the eigenvectors.

- For $\lambda_1 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\zeta}_1 = \mathbf{0}$, which is $\boldsymbol{\zeta}_1 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} x \\ x - 1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$.

Hence, the general solution is:

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(t e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} \right).$$

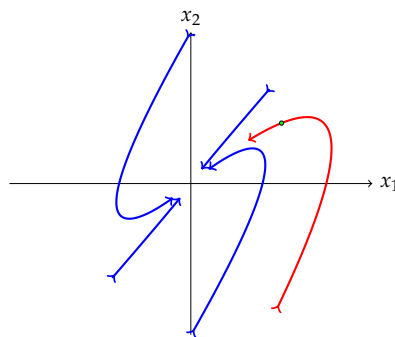
By the initial condition, we have $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, so:

$$\mathbf{x}(0) = \begin{pmatrix} C_1 + 0 \\ C_1 - C_2/4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore, $C_1 = 3$ and $C_2 = 4$, so the particular solution is:

$$\mathbf{x}(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t}.$$

The phase portrait looks as follows:



The initial value problem (marked green) has its trajectory marked in red.