PILOT Midterm 1 Review

Differential Equations

Johns Hopkins University

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As you prepare for midterm 1, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/SP26/`
  - Find the review problem sets for quiz 1.
  - Review the weekly problem sets 1 – 6.
- Review the *WebWork* provided by the instructor.
  - Check out the section review sheets from each TA.
- Join the PILOT Review Session. (You are here.)

Plan for today:

1. Go over all contents that we have covered for this semester so far.
2. In the end, we will open the poll to you. Please indicate which problems from the Review Set that you want us to go over.

Part 1:
Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

Preliminaries

- Classifications of Differential Equations
  - ODEs vs PDEs
- Modeling Using ODEs
  - Half Life Problem

When having various differential equations, we can classify them by their properties.

ODEs vs PDEs

Ordinary Differential Equations (ODEs) involves ordinary derivatives ($\frac{dy}{dt}$), while Partial Differential Equations (PDEs) involves partial derivatives ($\frac{\partial y}{\partial t}$).

This course focuses on ODEs, and it can also be classified in various different ways:

- **Single equation** involves one unknown and one equation, while **system of equations** involves multiple unknowns and multiple equations.
- The **order** of the differential equation is the order of the highest derivatives term.
- **Linear** differential equations has only linear dependent on the function, while **non-linear** differential equations has non-linear dependent on the function.

ODEs can be used for modeling. During modeling, it often follows the following steps:

1. Construction of the Models,
2. Analysis of the Models,
3. Comparison of the Models with Reality.

An example of modeling is the **half-life problem**.

Half Life Problem

The physics model for half life indicates the relationship between half life (τ) of a substance of amount $N(t)$ with initial amount N_0 at a time t is:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{\tau}},$$

where the rate of decay (λ) and half life (τ) are related by:

$$\tau \times \lambda = \log 2.$$

First Order ODEs

- Methods of Solving ODEs
  - Separable ODEs
  - Integrating Factor
- Existence and Uniqueness Theorems
- Autonomous ODEs
  - Rational Root Test
- Logistic Population Growth
  - Partial Fractions
- Exactness Problem
  - Integrating Factor for Non-Exact Case
- Bifurcation
  - Bifurcation Diagram

Here, we will introduce various ways of solving ODEs:

Separable ODEs

For ODEs in form $M(t) + N(y)\dfrac{dy}{dt} = 0$, it can be separated by:

$$M(t)dt + N(y)dy = 0.$$

When the ODE is not separable, we may consider using the **integrating factor**.

Integrating Factor

For ODEs in form $\dfrac{dy}{dt} + a(t)y = b(t)$, the integrating factor is:

$$\mu(t) = \exp\left(\int a(t)dt\right).$$

The existence and uniqueness for Initial Value Problem (IVP) tells us information on if the we can obtain a unique solution over some interval:

- For an IVP in simple form:

$$\begin{cases} \frac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$

  If $a(t)$ and $b(t)$ are continuous on an interval (α, β) and $t_0 \in (\alpha, \beta)$. Then, there exists a uniqueness solution y for (α, β) to the IVP.

Picard's Theorem

- For an IVP in general form:

$$\begin{cases} \dfrac{dy}{dt} = f(t,y), \\ y(t_0) = y_0. \end{cases}$$

For $t_0 \in I = (a,b)$, $y_0 \in J = (c,d)$, if $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous on interval $I \times J$. Then, there exists a unique solution on a smaller interval $I' \times J' \subset I \times J$, in which $(t_0, y_0) \in I' \times J'$.

Only Contrapositive is Guaranteed to be True

For both theorems, you can conclude that if *there does not exist a solution or the solution is not unique*, then *the conditions must not be satisfied*. You **cannot** conclude that if *the conditions are not satisfied*, then *there is no unique solution*.

Autonomous ODEs are in form of:

$$\frac{dy}{dt} = f(y).$$

The stability (stable/semi-stable/unstable) of equilibrium can be determined by phase lines, *i.e.*, the zeros of the function $f(t)$.

Rational Root Test

Let the polynomial with integer coefficients be defined as:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0,$$

then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p | a_0$ and $q | a_n$.

The logistic population growth model with population (y), growing rate (r), and carrying capacity (k) is given by:

$$\begin{cases} \frac{dy}{dt} = r\left(1 - \frac{y}{k}\right) y, \\ y(0) = y_0, \end{cases}$$

whose general solution is $y(t) = \dfrac{ky_0}{(k - y_0)e^{-rt} + y_0}$.

Partial Fractions

For a fraction in the form $\dfrac{C}{(x-a_1)^{n_1}(x-a_2)^{n_2}\cdots(x-a_m)^{n_m}}$, it can be decomposed in terms of:

$$\frac{C_{1,1}}{x-a_1} + \frac{C_{1,2}}{(x-a_1)^2} + \cdots + \frac{C_{1,n_1}}{(x-a_1)^{n_1}} + \cdots + \frac{C_{m,1}}{x-a_m} + \cdots + \frac{C_{m,n_m}}{(x-a_m)^{n_m}}.$$

The condition for a function in form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ to be exact is:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

For solving Exact ODEs, either finding $\int M(x,y)dx + h(y)$ or $\int N(x,y)dy + h(x)$ and taking partials again to fit gives the solution $\Psi(x,y) = C$.

Integrating Factor for Non-Exact Case

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N} dx\right) \text{ or } \mu(y) = \exp\left(\int \frac{N_x - M_y}{M} dy\right).$$

When a differential equation contains some unknown, fixed parameter C, its equilibriums would exhibit different behavior, the bifurcation value is the critical value such that the equilibriums have different stability.

Bifurcation Diagram

A bifurcation diagram is the vertical concatenation of phase portraits (C-y plot), in which the equilibriums will be marked for respective values of C.

Second Order ODEs

- Linear Homogeneous Cases
  - Complex Characteristic Roots
  - Repeated Characteristic Roots
- Linear Independence
  - Definition of Linearly Independence
  - Superposition Principle
- Reduction of Order
  - Product Rule and Chain Rule

Consider the linear homogeneous ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is $r^2 + pr + q = 0$, with real, distinct solutions r_1 and r_2, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t} \cos(\beta t) + c_2 e^{\lambda t} \sin(\beta t).$$

Repeated Characteristic Roots

If the solutions are repeated, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

To form a fundamental set of solutions, the solutions need to be linearly independent, in which the Wronskian (W) must be non-zero, meaning that:

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \cdots, f_n, \cdots\}$ is linearly independent when for $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots \in \mathbb{F}$ (typically $\mathbb{C}$):

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n + \cdots = 0 \iff \lambda_1 = \lambda_2 = \cdots = \lambda_n = \cdots = 0.$$

Superposition Principle

If $y_1(t)$ and $y_2(t)$ are solutions to $l[y] = 0$, then the solution $c_1 y_1(t) + c_2 y_2(t)$ are also solutions for all constants $c_1, c_2 \in \mathbb{R}$.

For non-linear second order homogeneous ODEs, when one solution $y_1(t)$ is given, the other solution is in form:

$$y_2(t) = u(t) \cdot y_1(t).$$

Product Rule and Chain Rule

- **Product Rule**: $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x)g(x) + f(x)\frac{dg}{dx}(x).$
- **Chain Rule**: $\frac{d}{dx}\left[f(g(x))\right] = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).$

Procedure of Reduction of Order

As long as $y_1(t)$ is a solution, you will be able to reduce the differential equation with respect to y_2 into a differential equation involving only $u''(t)$ and $u'(t)$ terms to solve for $\omega(t) = u'(t)$.

Part 2:
Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through selected problems from the practice problem set.
- We are also open to conceptual questions with the course.

Good luck on your first midterm.