

PILOT Midterm 2 Review

Differential Equations

Johns Hopkins University

Spring 2026

As you prepare for midterm 2, please consider the following resources:

- PILOT webpage for ODEs:
<https://jhu-ode-pilot.github.io/SP26/>
 - Find the review problem sets for midterm 2.
 - Review the weekly problem sets 7 – 11 (and spring break special set).
- Review the *WebWork* provided by the instructor.
 - Check out the section review sheets from each TA.
- Join the PILOT Review Session. (You are here.)

Plan for today:

- 1 Go over all contents that we have covered for this semester so far.
- 2 In the end, we will open the poll to you. Please indicate which problems from the Review Set that you want us to go over.

Part 1:

Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

1 Second Order ODEs (Continued)

2 Higher Order ODEs

3 System of First Order Linear ODEs

Second Order ODEs (Continued)

- Non-homogeneous Cases
 - Variation of Parameters
 - Undetermined Coefficients

Let the differential equation be:

$$Ay''(t) + By'(t) + Cy(t) = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$ and $y_2(t)$ be the two homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

Variation of Parameters

The particular solution of the differential equation can be written as the integrals of respective parts.

$$y_p = y_1(t) \int \frac{-y_2(t) \cdot g(t)}{W} dt + y_2(t) \int \frac{y_1(t) \cdot g(t)}{W} dt.$$

Another approach is less calculation intensive, but requires the function $g(t)$ to be constrained in certain forms.

Undetermined Coefficients

A guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$	Guess
Polynomials: $\sum_{i=0}^d a_i t^i$	$\sum_{i=0}^d C_i t^i$
Trig.: $\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \cos(ax)$
Exp.: e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

Higher Order ODEs

- Existence and Uniqueness Theorem
- Homogeneous Cases
 - Complex Characteristic Roots
 - Repeated Characteristic Roots
- Linear Independence
 - Definition of Linearly Independence
- Abel's Formula
- Non-Homogeneous Cases
 - Variation of Parameters
 - Undetermined Coefficients

For higher order IVP in form:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

If $P_0(t), P_1(t), \cdots, P_{n-1}(t)$, and $g(t)$ are continuous on an interval I containing t_0 . Then there exists a unique solution for $y(t)$ on I .

Only Contrapositive is Guaranteed to be True

Again, for this theorem, you can conclude that if *there does not exist a solution or the solution is not unique*, then *the conditions must not be satisfied*. You **cannot** conclude that if *the conditions are not satisfied*, then *there is no unique solution*.

The higher order homogeneous ODEs are in form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

By computing the characteristic equation

$r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$, with solutions r_1, r_2, \dots, r_n ,
the general solution is $y(t) = c_1e^{r_1t} + c_2e^{r_2t} + \cdots + c_ne^{r_nt}$.

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1e^{\lambda t} \cos(\beta t) + c_2e^{\lambda t} \sin(\beta t) + \text{rest of the solutions.}$$

Repeated Characteristic Roots

If the solutions are repeated with multiplicity m , the solution is:

$$y(t) = c_1e^{rt} + c_2te^{rt} + \cdots + c_mt^{m-1}e^{rt} + \text{rest of the solutions.}$$

To obtain the fundamental set of solutions, the Wronskian (W) must be non-zero, where Wronskian is:

$$W[y_1, y_2, \dots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \dots, f_n, \dots\}$ is linearly independent when for $\lambda_1, \lambda_2, \dots, \lambda_n, \dots \in \mathbb{F}$ (typically \mathbb{C}):

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n + \cdots = 0 \iff \lambda_1 = \lambda_2 = \cdots = \lambda_n = \cdots = 0.$$

For higher order ODEs in the form of:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

Its Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = Ce^{\int -P_{n-1}(t)dt},$$

where C is independent of t but depend on y_1, y_2, \cdots, y_n .

Let the differential equation be:

$$L[y^{(n)}(t), y^{(n-1)}(t), \dots, y(t)] = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t), y_2(t), \dots, y_n(t)$ be all homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

Variation of Parameters

The particular solution is:

$$y_p = y_1(t) \int \frac{W_1 g}{W} dt + y_2(t) \int \frac{W_2 g}{W} dt + \dots + y_n(t) \int \frac{W_n g}{W} dt,$$

where W_i is defined to be the Wronskian with the i -th column alternated into $(0 \ \dots \ 0 \ 1)^T$.

Undetermined Coefficients

Same as in degree 2, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$	Guess
Polynomials: $\sum_{i=0}^d a_i t^i$	$\sum_{i=0}^d C_i t^i$
Trig.: $\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \cos(ax)$
Exp.: e^{at}	Ce^{at}

Again, the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

System of First Order Linear ODEs

- Solving for Eigenvalues and Eigenvectors
- Linear Independence
 - Abel's Formula
- Phase Portraits
 - Node Graph

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - rI) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - rI) \cdot \boldsymbol{\zeta} = \mathbf{0}.$$

Suppose that the eigenvalues are distinct and the eigenvectors are linearly independent, the solution to the ODE is:

$$\mathbf{x} = c_1 \boldsymbol{\zeta}^{(1)} e^{r_1 t} + c_2 \boldsymbol{\zeta}^{(2)} e^{r_2 t} + \cdots + c_n \boldsymbol{\zeta}^{(n)} e^{r_n t}.$$

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det(\Psi(t)).$$

The system is linearly independent if the Wronskian is non-zero.

Abel's Formula

For the linear system in form:

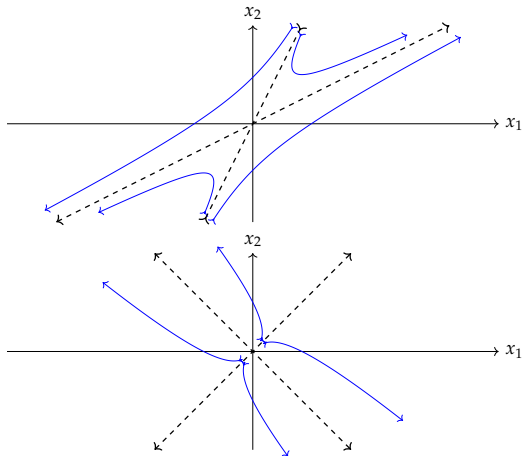
$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A , which is the sum of the diagonals, that is:

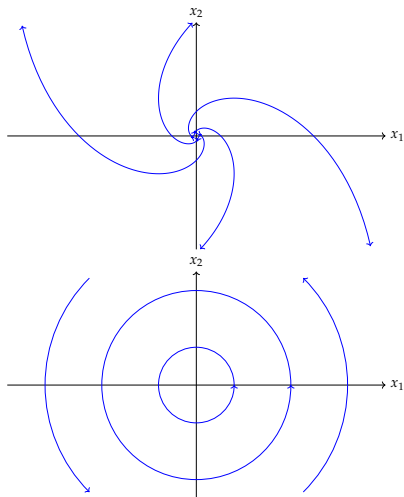
$$W = Ce^{\int \text{trace } A dt} = Ce^{\int (A_{1,1} + A_{2,2} + \dots + A_{n,n}) dt}.$$

In particular, we can sketch the linear system of \mathbb{R}^2 in terms of phase portraits given the eigenvalues and eigenvectors.

- For a node graph, we have it as (directions might vary):



- For a spiral/center graph, we have it as (directions might vary):



Part 2: Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through selected problems from the practice problem set.
- We are also open to conceptual questions with the course.

Good luck on your second midterm.