Additional Materials: Brief Review on Series

Differential Equations

Spring 2026

As we dive into fundamentals of mathematics, it is inevitable to encounter *sequences* and their sums. Discuss about the following sequences if they converge or not. If they converge, find the explicit sum.

(a)
$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

(b)
$$\sum_{k=0}^{\infty} \frac{1}{k!}.$$

(c)
$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!}.$$

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) Diligent readers should observe that $\sum_{k=0}^{\infty} 1/k$ is a harmonic series, hence it $\boxed{\text{diverges}}$. Otherwise, we can simply notice that:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k} &= \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}\cdots \\ &\geq \frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\cdots \\ &= \frac{1}{1}+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\cdots\right)+\cdots \\ &= \frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots,\end{aligned}$$

which diverges, hence our sequence $\sum_{k=0}^{\infty} 1/k$ must diverge.

(b) Here, we recall that the Taylor expansion of e^x at 0 is:

$$e^x = \sum_{k=0}^{\infty}\frac{1}{k!}e^0(x-0)^k = \sum_{k=0}^{\infty}\frac{1}{k!}x^k.$$

Evaluating the above equation at 1 gives that:

$$\sum_{k=0}^{\infty}\frac{1}{k!} = e^1 = \boxed{e},$$

in which the sequence $\boxed{\text{converges}}$.

(c) For this part, we want to note the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$ at 0 evaluated at $x = 1$ are, respectively:

$$\begin{array}{lcllllllll} e^1 &=& +\frac{1}{0!} & +\frac{1}{1!} & +\frac{1}{2!} & +\frac{1}{3!} & +\frac{1}{4!} & +\frac{1}{5!} & +\cdots \\ e^{-1} &=& +\frac{1}{0!} & -\frac{1}{1!} & +\frac{1}{2!} & -\frac{1}{3!} & +\frac{1}{4!} & -\frac{1}{5!} & +\cdots \\ \sin 1 &=& & +\frac{1}{1!} & & -\frac{1}{3!} & & +\frac{1}{5!} & -\cdots \\ \cos 1 &=& +\frac{1}{0!} & & -\frac{1}{2!} & & +\frac{1}{4!} & & -\cdots \end{array}$$

Since the first series converges, we know that the later three series converges *absolutely*, so we are free to move around terms. Thus comparing vertically gives us that:

$$\sum_{k=0}^{\infty}\frac{1}{(4k+1)!} = \boxed{\frac{e^1-e^{-1}}{4}+\frac{\sin 1}{2}}.$$

Specifically for the last part, the series has a pattern over terms that has a frequency of change for every 4 terms, so it is somewhat interesting to realize that $\sin x$ and $\cos x$ satisfy that they are the solutions to the differential equation whose 4-th derivative equals to itself (while earlier derivatives are not identical). We will explore more of these functions in the upcoming weeks.