Additional Material: Conceptual Eigenspace

Differential Equations

Spring 2026

Most likely in most ODE courses, you might be introduced to eigenvectors and eigenvalues purely by calculation approaches, but most students fails to know that eigenspace means in linear algebra. Alternatively, we will attempt to understand the concept of eigenspace through another approach.

To approach the purest understanding of eigenspace, let's forget everything about determinants.

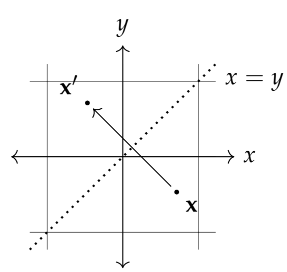
Definition. (Eigenvalue and Eigenvectors). Consider for a linear map T that maps V as a vector space to itself, denoted $T \in \mathcal{L}(V)$, the eigenvalue of T are values $\lambda \in \mathbb{F}$ (where $\mathbb{F}$ denotes the field) such that there exists some $v \in V$ such that:

$$T(v) = \lambda x,$$

and the corresponded elements (not necessarily vector) $v \in V$ are the eigenvectors.

Now, let's think about using the concept to find eigenvalues and eigenvectors without doing any matrix operation.

(a) Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as reflecting each point $(x, y) \in \mathbb{R}^2$ by the axis $x = y$, as of follows:



(i) What are the eigenvalues and eigenvectors of T.

(ii) Can you think of a way to represent T as a matrix? Feel free to check the eigenvalues and eigenvectors for the matrix align with what you get in part (i).

Now, we are going to go from the basic ideas of eigenvalues and eigenvectors to prove the following property on eigenvalues and eigenvectors.

Property. Given a matrix $A \in \mathbb{R}^{n\times n}$, the identity matrix $\mathrm{Id}_n \in \mathbb{R}^{n\times n}$, and constants $a, b \in \mathbb{R}$. Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of A and $\boldsymbol{\xi}$ is the associated eigenvector, then $b + a\lambda$ is an eigenvalue of $aA + b\,\mathrm{Id}_n$ and $\boldsymbol{\xi}$ is still the associated eigenvector.

(b) Prove the above property.

After this part, one should be justified on how the last part of the question works in the problem set. Hopefully, one can gain new understands with eigenspace.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) Let's think about this part without any technicality.

(i) Consider the reflection, and on the vector parallel perpendicular (or *orthogonal*) to the line $x = y$ would be able to preserve its direction (or at least it is parallel).
First, the eigenvalue would be 1, and the eigenvector is $(1, 1)$ (or any nonzero scalar multiples of that). Hence, the second eigenvalue would be -1, and the eigenvector is $(1, -1)$ (or any nonzero scalar multiples of that).

(ii) In fact, we can write T as a matrix in the following manner, for any vector parallel to $(1, 1)$, it gets mapped to $(1, 1)$, and vector parallel to $(1, -1)$, it gets mapped to $(-1, 1)$.
Therefore, the map maps $(0, 1) = \frac{(1,1)-(1,-1)}{2}$ to $\frac{(1,1)-(-1,1)}{2} = (1, 0)$ and maps $(1, 0) = \frac{(1,1)+(1,-1)}{2}$ to $\frac{(1,1)+(-1,1)}{2} = (0, 1)$. Hence, the matrix is:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b) The proof is trivial. Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of A and $\boldsymbol{\xi}$ is the associated eigenvector, then:

$$(aA + b\,\mathrm{Id}_n)(\boldsymbol{\xi}) = aA(\boldsymbol{\xi}) + b\,\mathrm{Id}_n(\boldsymbol{\xi}) = a(\lambda\boldsymbol{\xi}) + b\boldsymbol{\xi} = (a\lambda + b)\boldsymbol{\xi},$$

so $b + a\lambda$ is an eigenvalue of $aA + b\,\mathrm{Id}_n$ and $\boldsymbol{\xi}$ is still the associated eigenvector.