PILOT

Additional Material: A Putnam Problem

Differential Equations

Spring 2026

(*Putnam* 2023, Question 1.6). Determine the smallest positive real number r such that there exists differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying:

- $f(0) > 0$,
- $g(0) = 0$,
- $|f'(x)| \leq |g(x)|$ for all x,
- $|g'(x)| \leq |f(x)|$ for all x, and
- $f(r) = 0$.

You may give an answer *without* a rigorous proof, as the proof is out of scope of the course.
Hint: Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating f and g.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

Here, we first provide a “simplified” case, *i.e.*, we are constructing a dynamical system in which we pick equality for the inequality, that is:

$$\begin{cases} |f'(x)| = |g(x)|, \text{ and} \\ |g'(x)| = |f(x)|. \end{cases}$$

Without loss of generality, we may assume that f and g are non-negative before r, so the system becomes:

$$\begin{cases} f' = -g \\ g' = f \end{cases},$$

or equivalently, $\mathbf{y} = \begin{pmatrix} f \\ g \end{pmatrix}$ that $\mathbf{y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{y}$. Clearly, we observe the eigenvalues are $\pm \mathrm{i}$ as the polynomial is $\lambda^2 + 1 = 0$. Moreover, the eigenvectors for $\lambda_1 = \mathrm{i}$ is when $\begin{pmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$, in which we have $\boldsymbol{\xi} = y \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}$, and that solution is:

$$\mathbf{y} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} e^{\mathrm{i}x} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} (\cos x + \mathrm{i} \sin x) = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + \mathrm{i} \begin{pmatrix} \cos x \\ \sin x, \end{pmatrix}$$

and by conjugation, the solution should be:

$$\begin{pmatrix} f \\ g \end{pmatrix} = C_1 \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + C_2 \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}.$$

Note that with the given initial condition that $g(0) = 0$, this enforces $C_1 = 0$, thus $f(x) = C\cos x$ and $g(x) = C \sin x$, and we know that $f(r)$ is zero first at $r = \boxed{\pi/2}$.

The above version has some reasoning, but is not a rigorous proof at all, since this does not consider if r could be smaller than $\pi/2$. *For students with interests, we provide the complete proof from the Putnam competition from Victor Lie, as follows.*

Proof. Without loss of generality, we assume $f(x) > 0$ for all $x \in [0, r)$ as it is the first positive zero. By the fundamental theorem of calculus, we have:

$$|f'(x)| \le |g(x)| \le \left| \int_0^x g(s) ds \right| \le \int_0^x |g(s)| ds \le \int_0^t |f(s)| ds.$$

Now, as we denote $F(x) = \int_0^x f(s) ds$, we have:

$$f'(x) + F(x) \ge 0 \text{ for } x \in [0, r].$$

For the sake of contradiction, we suppose $r < \pi/2$, then we have:

$$f'(x) \cos x + F(x) \cos x \ge 0 \text{ for } x \in [0, r].$$

Notice that the left hand side is the derivative of $f(x) \cos x + F(x) \sin x$, so an integration on $[y, r]$ gives:

$$F(r) \sin r \ge f(y) \cos y + F(y) \sin(y).$$

With some rearranging, we can have:

$$F(r) \sin r \sec^2 y \ge f(y) \sec y + F(y) \sin y \sec^2 y$$

Again, we integrate both sides with respect to y on $[0, r]$, which gives:

$$F(r)\sin^2 r \geq F(r),$$

and this is impossible, so we have a contradiction.

Hence we must have $r \geq \pi/2$, and since we have noted the solution $f(x) = C\cos x$ and $g(x) = C\sin x$, we have proven that $r = \pi/2$ is the smallest case. $\square$