



Problem Set 12: Solutions

Differential Equations

Spring 2026

1. (System with Unknown Coefficients). Let a non-linear system for $x = x(t)$ and $y = y(t)$ be:

$$\begin{cases} x' = \alpha x - y + y^2, \\ y' = x + \alpha y. \end{cases}$$

(a) Show that $(0,0)$ is a critical point, and show system is locally linear at $(0,0)$ for all $\alpha \in \mathbb{R}$.

(b) Classify the critical point $(0,0)$ and sketch a few phase portraits of the linearized system.

Solution:

(a) *Proof.* To show that $(0,0)$ is a critical point, we just plug it into the system as:

$$\begin{cases} x'(0,0) = 0 - 0 + 0 = 0, \\ y'(0,0) = 0 + 0 = 0. \end{cases}$$

Hence $(0,0)$ is a critical point.

To consider the local linearity, we compute the Jacobian matrix as:

$$J[x', y'] = \begin{pmatrix} \partial_x x' & \partial_y x' \\ \partial_x y' & \partial_y y' \end{pmatrix} = \begin{pmatrix} \alpha & -1 + 2y \\ 1 & \alpha \end{pmatrix}.$$

When we evaluate it at $(0,0)$, we have:

$$\text{ev}_{(0,0)} J[x', y'] = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}.$$

We see that the determinant is $\alpha^2 + 1 > 0$, hence it is locally linear for all $\alpha \in \mathbb{R}$. □

(b) For the locally linear system, we classify the linear approximation as:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \sim \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, the characteristic equation is $(\alpha - \lambda)^2 + 1 = 0$, that is $\lambda = \alpha \pm i$.

Depending on different cases for α , we have different results:

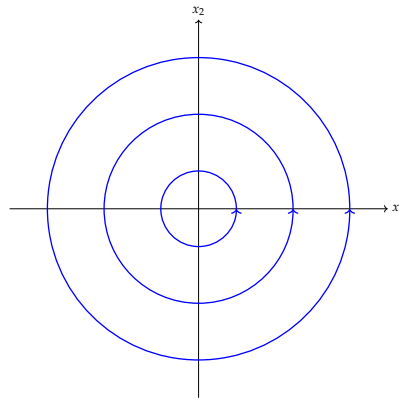
- When $\alpha = 0$, it is a stable center.
- When $\alpha > 0$, it is unstable spiral.
- When $\alpha < 0$, it is asymptotically stable spiral.

The graphs are on the next page.

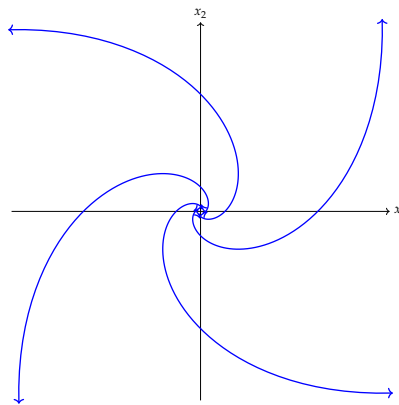
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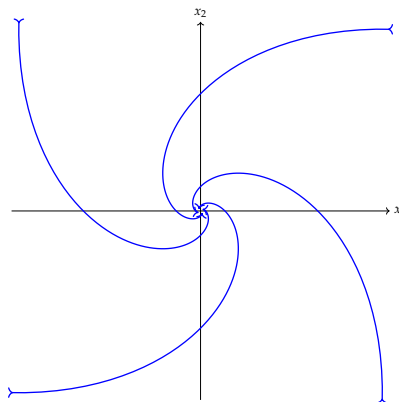
- For $\alpha = 0$:



- For $\alpha > 0$:



- For $\alpha < 0$:



2. (Phase Portraits for Repeated Roots). Find the solutions to the following linear system differential equation, sketch a few phase portraits, and classify its type and stability.

(a)
$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix} \cdot \mathbf{x}.$$

(b)
$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \cdot \mathbf{x}.$$

Solution:

(a) Again, we first find the eigenvalues as:

$$0 = \det \begin{pmatrix} 4 - \lambda & 2 \\ -2 & 0 - \lambda \end{pmatrix} = (4 - \lambda)(-\lambda) + 4 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

Hence, the eigenvalue is 2 (with *algebraic* multiplicity 2, see additional materials). Then, we look for the eigenvalue, that is ζ such that :

$$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \cdot \zeta = \mathbf{0}.$$

Hence, we have $2\zeta_1 + 2\zeta_2 = 0$, so $-\zeta_1 = \zeta_2$, which is $\zeta = (1, -1)$.

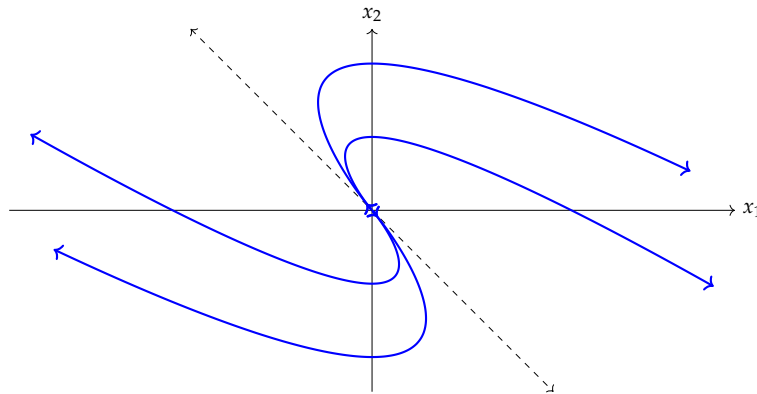
Then, we need to find the other solution with vector η such that:

$$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \cdot \eta = \zeta.$$

Hence, we have $2\eta_1 + 2\eta_2 = 1$, so $\eta = (1/2, 0)$. Hence, the solution is:

$$\mathbf{x} = \left[C_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \left(t e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right) \right].$$

Here, we may graph the phase portraits as:



Here, the graph has a unstable improper node.

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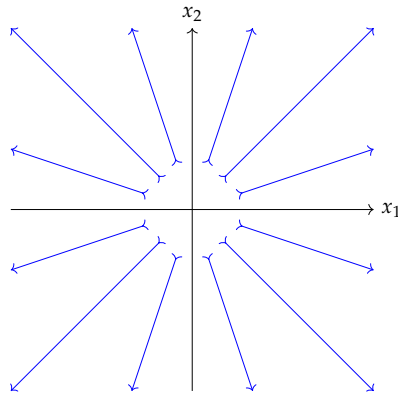
(b) Here, we find that the eigenvalue is 3 with *algebraic* multiplicity 2, and since it is diagonal, the *geometric* multiplicity is also 2. Here, we make this into a degenerated case:

$$\begin{cases} x_1' = 3x_1, \\ x_2' = 3x_2. \end{cases}$$

Hence, the solution is:

$$\begin{cases} x_1 = C_1 e^{3t}, \\ x_2 = C_2 e^{3t}. \end{cases}$$

Here, we may graph the phase portraits as:



Here, the graph has a unstable proper node.

3. (Critical Point). Find all the critical point in the following first order system:

$$\begin{cases} x' = 2x^3 - x^2 - 4x + 3 - y^2, \\ y' = 2x - y. \end{cases}$$

Solution:

Here, we set both equations as 0, so we have:

$$\begin{cases} 0 = 2x^3 - x^2 - 4x + 3 - y^2, \\ 0 = 2x - y. \end{cases}$$

Hence, we let the second equation be:

$$y = 2x,$$

and we plug it into the first equation to be:

$$0 = 2x^3 - x^2 - 4x + 3 - 4x^2 = 2x^3 - 5x^2 - 4x + 3.$$

Here, we attempt to factor the above by the rational root test (c.f. §1.3), the rational roots has to be one of $\pm 1, \pm 3, \pm 1/2, \pm 3/2$. We first note that $x = -1$ is a root, then we factor it into:

$$(2x^3 - 5x^2 - 4x + 3)/(x + 1) = 2x^2 - 7x + 3.$$

Thus, the last two roots are $1/2$ and 3 . Hence, the critical points are:

$$\boxed{(-1, -2), (1/2, 1), \text{ and } (3, 6)}.$$

4. (Nonlinear at origin). Let the linear system be:

$$\begin{cases} x' = y, \\ y' = x + 2x^3. \end{cases}$$

- (a) Show that the origin is a saddle point.
 (b) Sketch a phase portrait for the linearized system. Note that where all the trajectories of the linear system tend to the origin.

Solution:

- (a) *Proof.* Here, we first verify that it is critical point, that is $x'(0,0) = 0$ and $y'(0,0) = 0 + 0 = 0$. Then, we check the Jacobian matrix:

$$\text{ev}_{(0,0)} J[x', y'] = \text{ev}_{(0,0)} \begin{pmatrix} 0 & 1 \\ 1 + 6x^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose determinant is -1 , so the system is locally linear, so the linear system is:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, we have the characteristic equation as $\lambda^2 - 1 = 0$, so the roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. Thus, it is a unstable saddle point. \square

- (b) To sketch the diagram, we would want the eigenvectors.

- When $\lambda_1 = 1$, then we have $\zeta^{(1)}$ such that $-\zeta_1^{(1)} + \zeta_2^{(1)} = 0$, so the eigenvector is $(1, 1)$.
- When $\lambda_2 = -1$, then we have $\zeta^{(2)}$ such that $\zeta_1^{(1)} + \zeta_2^{(1)} = 0$, so the eigenvector is $(1, -1)$.

Hence, the diagram is as:

