Problem Set 12

Differential Equations

Spring 2026

As of right now, we have finished the chapter on first order linear system. As we are about to get into nonlinear system, let us review what we have learned about linear systems.

- Concepts:
  - Vector space & Eigenspace
  - Fundamental Matrix
  - Existence & Uniqueness Theorem
- Methods to solve linear systems of ODEs:
  - Eigenvectors & Eigenvalues
  - Repeated Roots
  - Phase Portraits and Stability

Clubs & Orgs Bulletin

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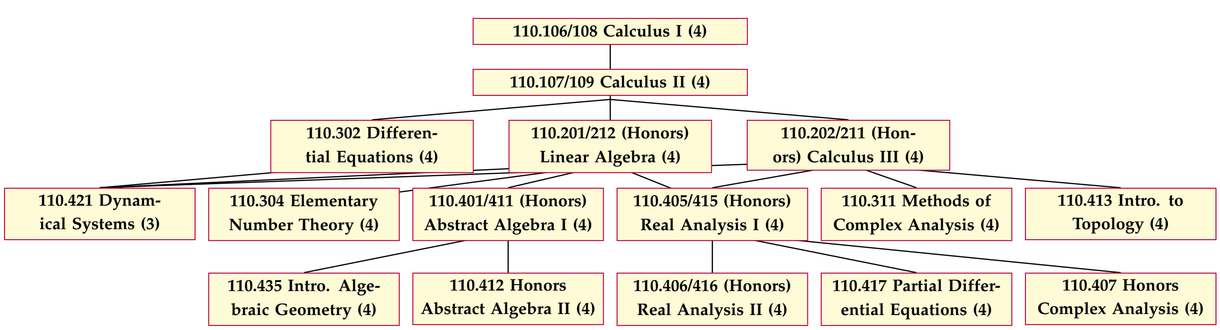
Tip of the Week

Class registration for fall 2026 is next week. Keep these dates in mind: April 13 for continuing & Rising Seniors, April 15 for Rising Juniors, and April 17 for Rising Sophomores.



1. (System with Unknown Coefficients). Let a non-linear system for $x = x(t)$ and $y = y(t)$ be:

$$\begin{cases} x' = \alpha x - y + y^2, \\ y' = x + \alpha y. \end{cases}$$

   (a) Show that $(0,0)$ is a critical point, and show system is locally linear at $(0,0)$ for all $\alpha \in \mathbb{R}$.

   (b) Classify the critical point $(0,0)$ and sketch a few phase portraits of the linearized system.

2. (Phase Portraits for Repeated Roots). Find the solutions to the following linear system differential equation, sketch a few phase portraits, and classify its type and stability.

   (a)
$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix} \cdot \mathbf{x}.$$

   (b)
$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \cdot \mathbf{x}.$$

3. (Critical Point). Find all the critical point in the following first order system:

$$\begin{cases} x' = 2x^3 - x^2 - 4x + 3 - y^2, \\ y' = 2x - y. \end{cases}$$

4. (Nonlinear at origin). Let the linear system be:

$$\begin{cases} x' = y, \\ y' = x + 2x^3. \end{cases}$$

(a) Show that the origin is a saddle point.

(b) Sketch a phase portrait for the linearized system. Note that where all the trajectories of the linear system tend to the origin.