Additional Material: Nonlinear System

Differential Equations

Spring 2026

Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:

$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$

Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.

(a) Find the set of all equilibrium(s) for $\mathbf{x}$.

(b) Find the set in which the equilibrium(s) is locally linear.

Now, $F : \mathbb{R}^2 \to \mathbb{R}$ is not necessarily well-behaved.

(c) Construct a function F such that $\mathbf{x}$ has a equilibrium that is <u>not</u> locally linear.
Hint: Consider the condition in which a non-linear system is locally linear.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) Here, we note that the equilibrium is when $F(\mathbf{x}) = 0$, *i.e.*, $\sin x_1 + \csc(3x_2) = 0$. Here, we note that the image of $\sin x_1$ is $[-1, 1]$ and the image of $\sec(3x_2)$ is $(-\infty, -1] \sqcup [1, \infty)$, this implies that $\sin x_1 + \sec(3x_2)$ is zero only if $\sin x_1 = \pm 1$ and $\sec(3x_2) = \mp 1$, correspondingly.
First, we consider the set in which x_1 is $+1$, that is:
$$\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$
Correspondingly, we consider the set in which x_2 is -1, that is:
$$\left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\}.$$
Then, we consider the set in which x_1 is -1, that is:
$$\left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\}.$$
Likewise, we consider the set in which x_2 is $+1$, that is:
$$\left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}.$$
Therefore, set theoretically, we have the set of all equilibriums as:
$$\boxed{\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}}.$$

(b) Here, one should notice that the Jacobian matrix would have the first row identical with the second row, so its determinant is consistently zero, and it is not locally linear at any value.

(c) Clearly, we must enforce that $F(\mathbf{x})$ is $\boxed{\text{not twice differentiable}}$ with some partial derivatives near the equilibrium point(s). One trivial example could be using the absolute value, such as $F(\mathbf{x}) = |x_1| + |x_2|$, where $(0, 0)$ is a equilibrium but it is not differentiable.
For capable readers, we invite them to look for more functions, such as the Weierstrass Function, a continuous function that is *nowhere* differentiable:
$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cos(3^k x).$$