PILOT

Problem Set 13

Differential Equations

Spring 2026

In the digressions to non-linear system, we are gradually seeing more dynamical systems and applications to the real life situations. As we explore cases with the non-linear system, please keep in mind that we are always trying to find a linear case to model as we zoom in. At the same moment, please take a moment to review what we have learned together.

- Concepts:
  - Nonlinear System
  - Locally Linear System
  - Jacobian Matrix
- Models:
  - Predator-Pray Model
  - Competing Species Model
  - Limit Cycles

Tip of the Week

Spring at Hopkins is headlined by the annual Spring Fair (April 2325, 2026), featuring a food marketplace, arts, games, fireworks, and live music. Other popular spring activities include attending Alumni Weekend events, exploring the nearby, blooming Sherwood Gardens, and engaging in outdoor activities on the Homewood campus quads.

1. (Limit Cycles). Determine the periodic solution, if there are any, of the following system:

$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \\ y' = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \end{cases}$$

2. (Modeling Politics). Suppose D and R are two parties on a non-existing country on the center of Mars. For the simplicity of this problem, they, *unfortunately*, have no elections. Therefore, we can model the amount of the supporter for each party (in millions), denoted x_D and x_R with the following relationship:

$$\begin{cases} \dfrac{dx_D}{dt} = x_D(1 - x_D - x_R), \\ \dfrac{dx_R}{dt} = x_R(3 - 2x_D - 4x_R). \end{cases}$$

Find all possible endings (say arbitrarily long after, that is $t \to \infty$) of the number of supporters (in millions) for the two parties.

3. (A Case Study on Tariffs). Suppose the tariff system in Mars (between all countries there) is based on the same formula, which is as follows:
$$\Delta\tau_i = \frac{x_i - m_i}{\varepsilon \times \varphi \times m_i}.$$
Here, $\Delta\tau_i$ means the change in tariff, x_i means the total import sale into your country from country i, m_i means the total export sale from your country to the country i, and $\varepsilon \times \varphi$ is 2.
Furthermore, a numerical estimation method in ODEs is called *Euler's Method*, and we will use the reverse of that to obtain an ODE model that:
$$\frac{d\tau(t)}{dt} \approxeq \frac{x(t) - m(t)}{2m(t)}.$$
Now, suppose there is another county, and you want to analyze the trends of tariffs with that country. With ϑ denoting their country's tariff on your country's import, we can create a system.
$$\begin{cases} \dfrac{d\tau(t)}{dt} = \dfrac{x(t) - m(t)}{2m(t)}, \\ \dfrac{d\vartheta(t)}{dt} = \dfrac{m(t) - x(t)}{2x(t)}. \end{cases}$$
For the simplicity of economics, we can model the import sale and export sale as:
$$x(t) = a - b\tau(t) \qquad \text{and} \qquad m(t) = c - d\vartheta(t),$$
where a, b, c, d are positive real constants.

(a) Write down the system of differential equations to model the tariffs as a vector $\mathbf{x}(t) = \big(\tau(t), \vartheta(t)\big)$.

(b) Find the set of all equilibrium points on this nonlinear system.

(c) Interpret some issues with the assumptions of this model.

4. (Dispersion of Heat). For this problem, we consider the dispersion of heat for an object in an environment with fixed temperature. Here, let $\theta := \theta(t)$ be the temperature of the object and θ_0 denote the fixed temperature of the environment, we may model the temperature of the object by:

$$\frac{d\theta}{dt} = -\frac{1}{\kappa}(\theta - \theta_0),$$

where κ is a fixed positive constant, representing the rate of heat dispersion.
Suppose that we have a rigid body of 100°C (equivalently 212°F), and the room temperature is fixed as 20°C (equivalently 68°F, and this is also condition for STP, standard temperature and pressure). Also, we assume that $\kappa = 2$.

   (a) Construct the differential equation for the above system.

   (b) Use *Euler's method* with step size of 1 to approximate the temperature at $t = 3$.

   (c) Identify if the approximation of temperature is an underestimate or an overestimate.