PILOT

Additional Materials: Lipschitz Continuity

Differential Equations

Spring 2026

Recall **Picard's Theorem** that you have seen over class:

> For an IVP in general form:
> $$\begin{cases} \dfrac{dy}{dt} = f(t,y), \\ y(t_0) = y_0. \end{cases}$$
> For $t_0 \in I = (a,b)$, $y_0 \in J = (c,d)$, if $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous on interval $I \times J$. Then, there exists a unique solution on a smaller interval $I' \times J' \subset I \times J$, in which $(t_0, y_0) \in I' \times J'$.

There exists a theorem called **Picard-Lindelöf theorem** that exhibits a weaker condition on the class of differential equations in which a solution is guaranteed to exist and is unique, namely the **Lipschitz continuity** condition. We will be exploring this topic this week.

First of all, we will provide you with the definition of **Lipschitz continuity**:

> A function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by $f(x,y)$ is Lipschitz continuous with respect to x if there exists some constant $K \geq 0$ such that for all $x_1, x_2 \in \mathbb{R}$ and for any $y \in \mathbb{R}^n$:
> $$|f(x_1,y) - f(x_2,y)| \leq K|x_1 - x_2|.$$

(a) Show that $f(x) = 2x$ is Lipschitz continuous and $g(x) = x^2$ is not Lipschitz continuous.

Then, we will be investigating how "strong" this Lipschitz continuity condition is, first lets compare it with continuous functions. In case you forget the definition of continuous functions, use the definition below:

> A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for every $x \in \mathbb{R}$, and for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x' \in \mathbb{R}$:
> $$|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon.$$

(b) Show that a Lipschitz continuity function is continuous.

Recall our original Picard's theorem, it was casting the condition of continuously differentiable, and we will now investigate this part.

(c) Given an example such that continuously differentiable does not imply Lipschitz continuity.

(d) Now, suppose the function is limited to a closed and bounded interval, continuously differentiable implies Lipschitz continuity. (*Hint: Use Extreme Value Theorem and Intermediate Value Theorem.*)

Now, we know that a closed and bounded region is necessary for this new theorem to work. Now, lets view the **Picard-Lindelöf theorem**:

Given an IVP:

$$\begin{cases} \dfrac{dy}{dt} = f(t, y), \\ y(t_0) = y_0. \end{cases}$$

with some closed rectangle $[a, b] \times [c, d]$ with the initial condition $(t_0, y_0) \in (a, b) \times (c, d)$, and f is continuous with respect to t and Lipschitz continuous with respect to y, then there exists a unique solution for the IVP on $[t_0 - \epsilon, t_0 + \epsilon]$ for some ϵ.

(e) Now, provide an IVP such that a unique solution can be guaranteed by the Picard-Lindelöf theorem, but not Picard's theorem.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) We closely abide to the definition, and it is not hard to notice that for $f(x) = 2x$, we trivially have:

$$|f(x_1) - f(x_2)| = 2|x_1 - x_2|,$$

which satisfies the Lipschitz continuous.
However, for $g(x) = x^2$, one should notice that for any $K \geq 0$, we have:

$$|f(0) - f(K+1)| = (K+1)^2 > K(K+1),$$

hence it is not Lipschitz continuous.

(b) This proof should be quite trivial in terms of definition, for any $\epsilon > 0$, since the function is Lipschitz, so there exists some constant $K \geq 0$ such that:

$$|f(x) - f(y)| \leq K|x - y|.$$

Therefore, let $\delta = \epsilon / K$, we then have for any $x, y \in \mathbb{R}$ such that $|x - y| < \delta$, we have:

$$|f(x) - f(y)| \leq K|x - y| < K \cdot \frac{\epsilon}{K} = \epsilon.$$

Therefore, we have shown that f is continuous.

(c) Using our previous example, $g(x) = x^2$, it is continuously differentiable (whose derivative is $2x$, and it is continuous), but we have shown that $g(x)$ is not Lipschitz.

(d) Note that we are given that f is continuously differentiable, then we know that on this closed and bounded interval, $|f'(x)|$ attains its maximum at some value $C := \max\{|f'(x)|\}$. Then, by the intermediate value theorem, for any x, y in this interval such that $x < y$, we have:

$$\frac{f(y) - f(x)}{y - x} \leq f'(x),$$

which implies that:

$$|f(y) - f(x)| \leq C|y - x|,$$

and thus f is Lipschitz.

By here, diligent readers should notice that over a closed and bounded region, the Lipschitz continuity is stronger than continuity but weaker than continuously differentiable, so it is a theorem that requires less condition than what was required in class.

(e) Almost immediately, we notice that function $f(x) = |x|$ is Lipschitz, but it is not continuously differentiable (discontinuity at 0 for $f'(x)$), so we can create an IVP as follows:

$$\begin{cases} y'(x) = |x|, \\ y(0) = 0. \end{cases}$$

In which we have a unique solution $y(x) = \frac{|x|x}{2}$, while it can be implied by the Picard-Lindelöf theorem, but not Picard's theorem.