PILOT

Problem Set 4

Differential Equations

Spring 2026

As we are familiarizing ourselves with more techniques of solving ODEs, we will also delve into some theory-intensive contents: investigating how certain criteria helps us to guarantee a local/global unique solution. It is important to acquaint yourself with the skills to solve equations quickly while also equip yourself with knowledge of how things work deep in the box.

Tip of the Week

There are multiple modes of free transportation to navigate around Baltimore. Use the TransLoc app to book the Blue Jay Shuttle, which runs between 6pm and 2am. The Collegetown Shuttle can be taken to Towson Center, and the Charm City Circulator stops at Mount Vernon and Penn Station. The JHMI shuttle is often used to reach the Johns Hopkins Hospital. Visit Homewood Student Affairs: Resources for Students for more information.

1. (Existence of Largest Interval). For the following IVPs, determine the largest interval in which a solution is guaranteed to exist from the *Existence and Uniqueness Theorem*.

   (a)
   $$\begin{cases} (t-3)y' + (\log t)y = 2t, \\ y(1) = 2. \end{cases}$$

   (b)
   $$\begin{cases} (4-t^2)y' + 2ty = 3t^2, \\ y(1) = -3. \end{cases}$$

   (c)
   $$\begin{cases} y' + (\tan t)y = \sin t, \\ y(\pi) = 0. \end{cases}$$

2. (Existence and Uniqueness for IVP). Suppose $f(x)$ is non-zero, let an initial value problem be:

$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

(a) Show that the differential equation is **not** linear.

For the next two parts, suppose $f(x) = \tan x$.

(b) State, without justification, the open interval(s) in which $f(x)$ is continuous.

(c) Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

(d) Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

3. (Bifurcation Diagram). For the first-order autonomous ODE:

$$\frac{dx}{dt} = x^2 - 2x + c,$$

with parameter $c \in \mathbb{R}$, do the following:

(a) Sketch all of the qualitatively different graphs of $f(x) = x^2 - 2x + c$, as c is varied.

(b) Determine any and all bifurcation values for the parameter c.

(c) Sketch a bifurcation diagram for this ODE.

4. (Make an Exact ODE). Let a differential equation on $y := y(x)$ be defined as follows:
$$xy^2 + bx^2y + (x+y)x^2y' = 0.$$
Suppose this differential equation is exact. Find the appropriate value of b and then solve for the solution of the differential equation.