Additional Material: Canonical Decomposition

Differential Equations

Spring 2026

Recall **Euler's theorem**, which states that for any $z \in \mathbb{C}$, we have:

$$\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z).$$

Introducing complex numbers is a very important inspiration to mathematics. Moreover, the complex numbers open up a new range for trigonometric functions.

(a) Express $\sin(z)$ and $\cos(z)$ in terms of exponential functions, where $z \in \mathbb{C}$ is a complex number.

Therefore, we will be able to extend the definition of sine and cosine functions to the complex domain with this method. In the area of complex analysis, there is an interesting theorem that would sound unreasonable for most if we consider complex numbers.

Liouville's Theorem. Every analytic function f for which there exists a positive number M such that:

$$|f(z)| \leq M \qquad \text{for all } z \in \mathbb{C},$$

then f is constant.

This theorem should sound unreasonable to many of people who are used to living in the real domain, and it is. For readers, you can assume analytic the same as definition in elementary calculus, which means it can be express as the convergence of polynomials.

(b) Show that $\sin(x)$ is a counterexample of Liouville's theorem when we consider the real input.

(c) Show that $\sin(z)$ is not a counterexample of Liouville's theorem when we consider the complex input.

Therefore, one should realize that we should reevaluate all the functions when we are in the domain of complex numbers. Then, we will study a famous example of decomposing the function for Euler's theorem.

(d) Given a function $\varphi\colon \mathbb{R} \to \mathbb{C}$ defined as $\varphi(x) = \exp(\mathrm{i}x)$. We can decompose $\varphi = i_\varphi \circ \tilde{\varphi} \circ \pi_\sim$, where $\pi_\sim$ is surjective, i_φ is injective, and $\tilde{\varphi}$ is bijective, which can be expressed as follows:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, two heads, "\pi_\sim"'] \arrow[rrr, bend left, "\varphi"] & X \arrow[r, hook, two heads, "\sim", "\tilde{\varphi}"'] & Y \arrow[r, hook, "i_\varphi"'] & \mathbb{C},
\end{tikzcd}$$

Find X and Y in the above commutative diagram.

Hint: Consider $\pi_\sim$ as a projection to an equivalent class, $\tilde{\varphi}$ as a modification of φ, and i_f as a map from the image to the co-domain.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) Here, readers shall recall that sine is odd and cosine is even. Now, we shall utilize this property with the exponentials, namely:

$$\exp(-\mathrm{i}z) = \cos(-z) + \mathrm{i}\sin(-z) = \cos(z) - \mathrm{i}\sin(z).$$

Combining this with the $\exp(\mathrm{i}z)$ expression, we may find that:

$$\sin(z) = \boxed{\frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}} \text{ and } \cos(z) = \boxed{\frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2}}.$$

(b) The sine function is definitely a counterexample. Consider that $-1 \leq \sin(x) \leq 1$ for all $x \in \mathbb{R}$ so $|\sin x| \leq 1$ for all $x \in \mathbb{R}$. Moreover, one should recall that $\sin(x)$ can be decomposed into a Taylor series, so it is analytic. Therefore, $\sin x$ is a counterexample for Liouville's Theorem when x is limited to be real.

(c) When we consider complex numbers, the story changes! Still $\sin z$ is analytic, but it would not be bounded. Recall that we have:

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2\mathrm{i}},$$

and for any positive real number $x > 1$, we have:

$$\sin(\mathrm{i}x) = \frac{\exp(-x) - \exp(x)}{2\mathrm{i}},$$

in which if we take the absolute value, we have $|\sin(\mathrm{i}x)| = \frac{|\exp(-x) - \exp(x)|}{2} \geq \frac{\exp x - \exp(-x)}{2}$, which could tend to ∞ as $x \to \infty$. Therefore, $\sin z$ is not bounded in terms of complex input.

(d) Here, lets discuss the motivation of this first, we want to decompose our function of several pieces, in which each piece has some unique properties, note that from Euler's formula, we see repetitive patterns in every 2π rotation, so our intention is to quotient out this triviality, so we have:

$$X = \boxed{\mathbb{R}/2\pi} := \big\{\{x + 2k\pi : k \in \mathbb{Z}\} : x \in \mathbb{R}\big\},$$

while the $\pi_\sim$ map will be a simple projection that projects $\mathbb{R}$ to its equivalent classes in $\mathbb{R}/2\pi$.
Since i_φ is injective, so we want Y to be the image of φ, so:

$$Y = \boxed{\mathbb{D}_1(0)} := \{x \in \mathbb{C} : |x| = 1\},$$

with the corresponding i_φ being the identity map.
Eventually, we now see $\tilde{\varphi}$ is bijective, since it is mapped to the image of φ and the injective part is enforced by having $\exp(z)$ and $\exp(z')$ being unique up to $z \sim z'$ in the equivalent class.

If you have taken abstract algebra or basic category theory, this is called a canonical decomposition, so we can think of more properties of each respective map and use various universal properties. Please check on some of these text to learn more about such decomposition.