Additional Material: Introduction to $L^2([0, 2\pi])$ Space

Differential Equations

Spring 2026

Recall that we have defined linear independence of functions, as follows:

Definition. (Linearly Independence).
Two functions f and g are *linearly independent* if $\lambda_1 f + \lambda_2 g = 0$ implies $\lambda_1 = \lambda_2 = 0$.

However, we can also a special class of functions known to be real-valued, "square-integrable" functions over $[0, 2\pi]$, which is called the $L^2([0, 2\pi])$ space:

Definition. ($L^2([0, 2\pi])$ Space).
A real-valued function f is in the $L^2([0, 2\pi])$ space if:

$$\int_0^{2\pi} |f(x)|^2 dx < +\infty.$$

(a) Show that $\sin x$ and $\cos x$ are in the $L^2([0, 2\pi])$ space.

Then, we can define *orthogonality* of two real-valued, "square-integrable" functions over $[0, 2\pi]$, f and g, as:

$$\int_0^{2\pi} f(x)g(x)dx = 0.$$

(b) Show that the set $\{\sin x, \cos x\}$ is linearly independent and orthogonal.

(c) Show that if $\{f(x), g(x)\}$ is orthogonal, then $C_1 f(x)$ and $C_2 g(x)$ is orthogonal.

(d) Note that $\{x, x^2\}$ are linearly independent, construct a basis that is orthogonal.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) This proof should be trivial, we have:

$$\int_0^{2\pi} |\sin x|^2 dx = \int_0^{2\pi} \sin^2 x = \int_0^{2\pi} \frac{1-\cos 2x}{2} = \frac{x}{2} - \frac{\sin 2x}{4}\Big|_0^{2\pi} = \pi < +\infty,$$

$$\int_0^{2\pi} |\cos x|^2 dx = \int_0^{2\pi} \cos^2 x = \int_0^{2\pi} \frac{1+\cos 2x}{2} = \frac{x}{2} + \frac{\sin 2x}{4}\Big|_0^{2\pi} = \pi < +\infty.$$

Therefore, we have $\sin x, \cos x \in L^2([0,2\pi])$.

(b) To show linear independence, we can compute the Wronskian as:

$$\mathrm{W}(\sin x, \cos x) = \det \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

Then, to show orthogonality, we have:

$$\int_0^{2\pi} \sin x \cos x dx = \frac{1}{2}\int_0^{2\pi} \sin(2x) dx = \frac{1}{2}\left[-\frac{1}{2}\cos(2x)\right]_0^{2\pi} = \frac{1}{4}\big(\cos 0 - \cos(4\pi)\big) = 0,$$

hence we have shown linear independence and orthogonality.

(c) By orthogonality, we have $\int_0^{2\pi} f(x)g(x)dx = 0$, so we have:

$$\int_0^{2\pi} C_1 f(x) \cdot C_2 g(x) dx = C_1 C_2 \int_0^{2\pi} f(x)g(x)dx = C_1 C_2 \cdot 0 = 0.$$

Hence orthogonality is preserved with scalar multiplications.

(d) The check of x and x^2 being linearly independent can be verified by Wronskian, and we leave this check to the readers. By the principle of superposition, we want to construct the second argument as $x^2 - Ax$, where A is a constant, now we take the inner product as:

$$\int_0^{2\pi} x(x^2 - Ax)dx = \int_0^{2\pi} (x^3 - Ax^2)dx = \frac{x^4}{4} - \frac{Ax^3}{3}\Big|_0^{2\pi} = 4\pi^4 - \frac{8A\pi^3}{3} = 0,$$

which forces A to be $3\pi/2$, so the orthogonal basis is now:

$$\boxed{\left\{x, x^2 - \frac{3\pi x}{2}\right\}}.$$

Diligent readers should notice that we have somehow constructed a "vector space" with a proper inner product. In fact, this space $L^2([0,2\pi])$ is considered a Hilbert Space, that is a infinite dimensional vector space with completeness and denseness. The $L^2([0,2\pi])$ is closely related to Fourier series, that has inarguable impacts on mathematics as well as sciences and engineering disciplines.