PILOT

Problem Set 6

Differential Equations

Spring 2026

Welcome to the realm of second order differential equations. As we just got into this part, we will explore the basic cases as well as some foundations of the second order differential equations. For this problem set, we will briefly investigate two core ideas, the linear independence and Euler's theorem.

Tip of the Week

Have you used Handshake to find jobs and internships at or beyond Hopkins? Handshake offers career workshops, recruitment information sessions, and job opportunities that target Johns Hopkins students and alumni. Setting up your Handshake profile can also make you visible to more recruiters and employers. It is a great professional resource designed to connect students with opportunities in their fields of interest: https://app.joinhandshake.com/.

1. (Second Order IVP with Dirichlet Condition). Given following initial value problem.

$$\begin{cases} y'' = -y, \\ y(0) = 0, \qquad y(b) = 0, \end{cases}$$

where $b \in \mathbb{R}$ and $b \neq 0$.

(a) Find the general solution to $y'' = -y$.

(b) Suppose $b = \frac{\pi}{4}$, find the solution.

(c) Give a value of b such that there is not a unique solution.

2. (Reduction of Order). Let a ODE be defined as follows:

$$t^2 y'' + 2ty' = 2y,\ t > 0.$$

Given a solution is $y_1(t) = t$, find the other solution by assuming $y_2(t) = u(t) \cdot y_1(t)$.

3. (Reduction of Order or Integrating Method). Let a differential equation be:

$$y''(t) + \frac{2}{t}y'(t) = 0.$$

(a) Verify that $y(t) = 1/t$ is one solution, then find a full set of solution.

(b) Consider $\omega(t) = y'(t)$, solve the differential equation by using integrating factor.

(c) Verify that the two methods give you the same set of the solutions.

4. (A Criterion on Linearly Independence). Recall that for the complex characteristic roots $x = \lambda \pm \mathrm{i}\mu$, the corresponding solutions are:

$$y_1 = e^{\lambda x}\sin(\mu x) \qquad \text{and} \qquad y_2 = e^{\lambda x}\cos(\mu x).$$

Of course, you may compute the Wronskian of $\mathrm{W}(e^{\lambda x}\sin(\mu x), e^{\lambda x}\cos(\mu x))$, which involves taking derivative with chain rule and product rule, and a lot of computation. Another approach to show linear independence is by returning to its definition.

> **Definition**. (Linearly Independence).
> Two functions f and g are *linearly independent* if $\lambda_1 f + \lambda_2 g = 0$ implies $\lambda_1 = \lambda_2 = 0$.

(a) Show that $y_1 = x$ and $y_2 = x^2$ is linearly independent via both approach.

(b) Show that $y_1 = e^{\lambda x}\sin(\mu x)$ and $y_2 = e^{\lambda x}\cos(\mu x)$ are linearly independent by using the definition.

(c) Let two functions be defined as follows:

$$y_1(x) = \mathbb{1}_{[0,1]}(x) = \begin{cases} 1, & \text{when } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{and} \qquad y_2 = \mathbb{1}_{[2,3]}(x) = \begin{cases} 1, & \text{when } 2 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the two functions are linearly independent.