Additional Material: Deeper Topics into $L^2([0,2\pi])$ Space

Differential Equations

Spring 2026

Recall from last week, we have defined a "vector space" of functions for $L^2([0,2\pi])$. In fact, this is how Fourier series is being defined. Here, $\{\sin(nx), \cos(nx)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis of $L^2([0,2\pi])$ space. To better understand how this relates to linear algebra, we will explore the following problems:

(a) Verify that $\{\sin(nx), \cos(nx), 1\}_{n\in\mathbb{Z}^+}$ is an orthogonal set.

Note that the verification of it being a basis is, in fact, much more complicated, so we will just bear with that. However, for any function $f \in L^2([0,2\pi])$, it is defined such that:

$$\int_0^{2\pi} \big(f(x)\big)^2 dx < +\infty.$$

(b) Verify that $f(x) = x$ is a $L^2([0,2\pi])$ function.

(c) Decompose $f(x) = x$ into sine and cosine functions, this is a Fourier series of $f(x) = x$.

At this moment, some of the readers might question: "What are we doing this all for?" The construction of a "vector space" is a very abstract approach. Of course, you will be able to construct to approximate all L^2 functions by a infinite sum. But the following concrete problem in PDEs could be solved by this assumption whereas we have obtained all the components, now, to solve this problem.

(d) The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length L, the displacement of string at $x \in [0, L]$ at time $t \in [0,\infty)$ is described as the function $u = u(x,t)$:

$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0,L) \text{ and } t \in [0,\infty); \\ \text{Initial Conditions:} & u(x,0) = \sin\left(\dfrac{2\pi x}{L}\right), & \\ & \dfrac{\partial u}{\partial t}(x,0) = \sin\left(\dfrac{5\pi x}{L}\right), & \text{where } x \in [0,L]; \\ \text{Boundary Conditions:} & u(0,t) = u(L,t) = 0, & \text{where } t \in [0,\infty); \end{cases}$$

where c is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*, $u(x,t) = v(x) \cdot w(t)$, and attempt to obtain a general solution that is *non-trivial*.
Hint: Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L), 1\}_{n\in\mathbb{Z}^+}$ forms an orthogonal basis.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) We just need to discuss a few cases separately.

- Case for $\sin(nx)$ and $\sin(mx)$, where $n \neq m$:
$$\begin{aligned}\int_0^{2\pi} \sin(nx)\sin(mx)\,dx &= \frac{1}{2}\int_0^{2\pi}\left[\cos((n-m)x) - \cos((n+m)x)\right]dx \\ &= \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x) - \frac{1}{n+m}\sin((n+m)x)\right]_{x=0}^{x=2\pi} = 0.\end{aligned}$$
- Case for $\sin(nx)$ and $\cos(mx)$:
$$\begin{aligned}\int_0^{2\pi} \sin(nx)\cos(mx)\,dx &= \frac{1}{2}\int_0^{2\pi}\left[\sin((n+m)x) + \sin((n-m)x)\right]dx \\ &= \frac{1}{2}\left[-\frac{1}{n+m}\cos((n+m)x) - \frac{1}{n-m}\cos((n-m)x)\right]_{x=0}^{x=2\pi} = 0.\end{aligned}$$
- Case for $\cos(nx)$ and $\cos(mx)$, where $n \neq m$:
$$\begin{aligned}\int_0^{2\pi} \cos(nx)\cos(mx)\,dx &= \frac{1}{2}\int_0^{2\pi}\left[\cos((n-m)x) + \cos((n+m)x)\right]dx \\ &= \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x) + \frac{1}{n+m}\sin((n+m)x)\right]_{x=0}^{x=2\pi} = 0.\end{aligned}$$

Hence, we have verified that this is a orthogonal set. Note that if you want an orthonormal set, you can easily normalize it via:
$$\left\{\frac{\sin(nx/2)}{\sqrt{\pi}}, \frac{\cos(nx/2)}{\sqrt{\pi}}\right\}_{n\in\mathbb{Z}^+}.$$

(b) This is because:

- Case for $\sin(nx)$ and $\sin(nx)$:
$$\int_0^{2\pi} \sin(nx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi}\left[1-\cos(2nx)\right]dx = \frac{1}{2}\left[x - \frac{1}{2n}\sin(2nx)\right]_{x=0}^{x=2\pi} = \pi.$$
- Case for $\cos(nx)$ and $\cos(nx)$:
$$\int_0^{2\pi} \cos(nx)\cos(nx)\,dx = \frac{1}{2}\int_0^{2\pi}\left[1+\cos(2nx)\right]dx = \frac{1}{2}\left[x + \frac{1}{2n}\sin(2nx)\right]_{x=0}^{x=2\pi} = \pi.$$

In fact, one can prove that this is in fact a basis, *i.e.*, it spans the whole $L^2([0,2\pi])$ space. There have been proofs by using Poisson kernel, complex analysis, or by the convergence of Fourier series. We will leave this as an interests for who might get interested in this.

(b) The verification that $f(x) = x$ is trivial:
$$\int_0^{2\pi} (f(x))^2 dx = \int_0^{2\pi} x^2 dx = \frac{1}{3}x^3\Big|_{x=0}^{x=2\pi} = \frac{1}{3}\cdot 8\pi = \frac{8}{3}\pi < +\infty.$$
Hence, we have verified that $f(x) = x$ is $L^2([0,2\pi])$.

(c) Note that we have the orthogonal basis, so we can think about project $f(x) = x$ to each basis:

- Project x to $\sin(nx)$ for $n \in \mathbb{Z}^+$:

$$\begin{aligned}
\mathrm{proj}_{\sin(nx)}(x) &= \frac{\langle \sin(nx), x\rangle}{\langle \sin(nx), \sin(nx)\rangle} \sin(nx) \\
&= \frac{\int_0^{2\pi} x \sin(nx) dx}{\pi} \sin(nx) \\
&= \left[-\frac{1}{n} x \cos(nx) + \frac{1}{n} \int \cos(nx)\, dx \right]_{x=0}^{x=2\pi} \frac{\sin(nx)}{\pi} \\
&= \left[-\frac{2\pi}{n} \cos(2n\pi) + 0 \right] \cdot \frac{1}{\pi} \sin(nx) = -\frac{2}{n} \sin(nx).
\end{aligned}$$

- Project x to $\cos(nx)$ for $n \in \mathbb{Z}^+$:

$$\begin{aligned}
\mathrm{proj}_{\cos(nx)}(x) &= \frac{\langle \cos(nx), x\rangle}{\langle \cos(nx), \cos(nx)\rangle} \cos(nx) \\
&= \frac{\int_0^{2\pi} x \cos(nx) dx}{\pi} \cos(nx) \\
&= \left[\frac{1}{n} x \sin(nx) - \frac{1}{n} \int \sin(nx)\, dx \right]_{x=0}^{x=2\pi} \frac{\cos(nx)}{\pi} = 0.
\end{aligned}$$

- Project x to 1:

$$\mathrm{proj}_1(x) = \frac{\langle 1, x\rangle}{\langle 1, 1\rangle} \cdot 1 = \frac{\int_0^{2\pi} x dx}{\int_0^{2\pi} 1 dx} = \frac{\frac{x^2}{2}\Big|_{x=0}^{x=2\pi}}{2\pi} = \pi.$$

Hence, the Fourier series of x on $[0, 2\pi]$ is:

$$f(x) = \boxed{\pi - \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)}.$$

(d) With the method of separation, we insert the separations back to the system of equation to obtain:

$$v(x) w''(t) = c^2 v''(x) w(t).$$

Now, we apply the separation and set the common ratio to be λ:

$$\frac{v''(x)}{v(x)} = \frac{1}{c^2} \cdot \frac{w''(t)}{w(t)} = \lambda.$$

Reformatting the boundary condition gives use the following initial value problem:

$$\begin{cases} v''(x) - \lambda v(x) = 0, \\ v(0) = v(L) = 0. \end{cases}$$

As a second order linear ordinary differential equation, we discuss all following cases:

- If $\lambda = 0$, then $v(x) = a + Bx$ and by the initial condition, $A = B = 0$, which gives the trivial solution, *i.e.*, $v(x) = 0$;
- If $\lambda = \mu^2 > 0$, then we have $v(x) = Ae^{-\mu x} + Be^{\mu x}$ and again giving that $A = B = 0$, or the trivial solution;
- Eventually, if $\lambda = -\mu^2 < 0$, then we have the solution as:

$$v(x) = A \sin(\mu x) + B \cos(\mu x),$$

and the initial conditions gives us that:

$$\begin{cases} v(0) = B = 0, \\ v(L) = A \sin(\mu L) + B \cos(\mu L) = 0, \end{cases}$$

where A is arbitrary, $B = 0$, and $\mu L = m\pi$ positive integer m.

Overall, the only non-trivial solution would be:

$$v_m(x) = A\sin(\mu_m x), \text{ where } \mu_m = \frac{m\pi}{L}.$$

Eventually, by inserting back $\lambda = -\mu_m^2$, we have $\lambda = -m^2\pi^2/L^2$, giving the solution to $w_m(t)$, another second order linear ordinary differential equation, as:

$$w_m(t) = C\cos(\mu_m ct) + D\sin(\mu_m ct), \text{ with } C, D \in \mathbb{R}.$$

By the *principle of superposition*, we can have our solution in the form:

$$u(x,t) = \sum_{m=1}^{\infty} [a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)] \sin(\mu_m x),$$

where our coefficients a_m and b_m have to be chosen to satisfy the initial conditions for $x \in [0, L]$:

$$u(x,0) = \sum_{m=1}^{\infty} a_m \sin(\mu_m x) = \sin\left(\frac{2\pi x}{L}\right),$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} c\mu_m b_m \sin(\mu_m x) = \sin\left(\frac{5\pi x}{L}\right).$$

Since we are hinted that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis, we now know that except for the following:

$$a_2 = 1 \text{ and } c\mu_5 b_5 = 1,$$

all the other coefficients are zero, so we have:

$$u(x,t) = \boxed{\cos\left(\frac{2\pi ct}{L}\right)\sin\left(\frac{2\pi x}{L}\right) + \frac{L}{5\pi c}\sin\left(\frac{5\pi ct}{L}\right)\sin\left(\frac{5\pi x}{L}\right)}.$$