Additional Material: Quotient Space

Differential Equations

Spring 2026

A few weeks ago, we have developed the concept that we can have the space of the functions to form a vector space, say $L^2([0, 2\pi])$ as a vector space.

More specifically, before moving to the nonhomogeneous differential equations, we have noticed that the solutions homogeneous differential equations form a subspace of the vector space, which is itself a vector space. As a quick reminder, here are some criteria that we can check a subset of a vector space being a subspace:

Theorem. Criteria for Subspace.
Suppose V is a vector space and $W \subset V$. If W satisfies the following, W is a subspace of V:

- Zero is included: $0 \in W$,
- Addition is closed: for all $w_1, w_2 \in W$, $w_1 + w_2 \in W$.
- Scalar multiplication is closed: for all $w \in W$ and $\lambda \in \mathbb{F}$, $\lambda w \in W$.

(a) Let y_1, y_2 be linearly independent solutions to a homogeneous second order differential equation, is the general solutions of the differential equation forming a subspace?

(b) If y_1 and y_2 is not linearly independent, are they still forming a subspace? If so, what can be its dimension?

(c) Show that the general solution to a nonhomogeneous differential equation does not form a subspace. Specifically, which criteria is violated?

Since the solution space to the nonhomogenous differential equation is not a subspace, we can introduce a new structure in vector spaces, *i.e.*, the **quotient** spaces.

Definition. Quotient Space.
Given an equivalence relationship $\sim$, we can define a quotient space over a vector space V as $V/\sim$ as:

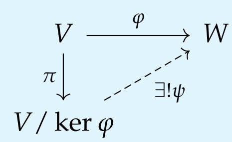
$$V/\sim = \{[x] := \{v \in V : v \sim x\}\},$$

i.e., the collection of all equivalence classes under the equivalence relation.

(d) Given the particular solution y_p, what should the equivalence space be, and what is the equivalence class for the solution to the nonhomogeneous ODE?

Eventually, we can get to a higher level result of such space, which is called a **universal property**.

Theorem. Universal Property for Quotient.
Given a equivalence relationship that is the kernel of a map φ, there exists a unique homomorphism (map) from the quotient space to the target space, portrayed as follows:



(e) For the case of nonhomogeneous solutions to a second order differential equation, how can you correspond the universal property for quotient?

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) Yes, it is forming a subspace, recall the solutions form that:

$$W = \{C_1y_1 + C_2y_2 : C_1, C_2 \in \mathbb{R}\}.$$

Here, one can check that:

- $0 \in W$ which is when $C_1 = C_2 = 0$,
- Addition is closed, since for any $w_1 = \lambda_1 y_1 + \lambda_2 y_2$ and $w_2 = \beta_1 y_1 + \beta_2 y_2$, we have:
$$w_1 + w_2 = (\lambda_1 + \beta_1)y_1 + (\lambda_2 + \beta_2)y_2 \in W.$$
- Scalar multiplication is closed, since for any $w = \lambda_1 y_1 + \lambda_2 y_2$ and $\lambda \in \mathbb{R}$, we have:
$$\lambda w = (\lambda\lambda_1)y_1 + (\lambda\lambda_2)y_2 \in W.$$

Specifically, this can be written as $\text{span}\{y_1, y_2\}$.

(b) When y_1 and y_2 are not linearly independent, $\text{span}\{y_1, y_2\}$ is still a subspace.
However, its dimension is actually dependent on the cases:

- If $y_1 = y_2 = 0$, we have $\text{span}\{y_1, y_2\} = \{0\}$, which has dimension 0.
- If $y_1 \neq 0$ or $y_2 \neq 0$, we have the span as the either nonzero term, so we have dimension 1.

(c) For the nonhomogeneous solution, it can be written as:

$$W = \{C_1y_1 + C_2y_2 + y_p : C_1, C_2 \in \mathbb{R}\},$$

here we should note that $y_p = 0$ and $y_p \notin \text{span}\{y_1, y_2\}$, so we have:

$$0 \notin W,$$

so it is not a subspace.
Note: You can also argue that addition / scalar multiplication is not closed as we can have $0 = y_1 - y_1 = 0 \cdot y_1 \notin W$ as well.

(d) Yes, we can form a quotient space, we define the equivalence relationship as:

$$x_1 \sim x_2 \text{ when } x_1 - x_2 \in \text{span}\{y_1, y_2\},$$

and we have the solution as:

$$[y_p]_\sim := \{x + y_p : x \in \text{span}\{y_1, y_2\}\} \in L^2([0, 2\pi])/\sim.$$

(e) For the problem, we can consider the following assignment:

- φ is a linear operator on y, y', y'' (assuming second order). For example, for the differential equation $L[y, y', y''] = y'' + 2y' + y$, $\varphi(y) = y'' + 2y' + y$.
- V is the space of twice differentiable functions.
- W can be the space of all functions.
- $\ker\varphi$ will correspond to the case when the nonhomogeneous part is zero, *i.e.*, the class of functions that satisfies $\varphi(y) = L[y, y', y''] = 0$, which is the subspace of the homogeneous case.
- Therefore, when we take the quotient space of these candidate solutions, it corresponds to a single function ψ, which maps each class of functions that leads to general solutions to a second order ODE with the right hand side specifically as the nonhomogeneous (or potentially homogeneous) solution, and such relationship is unique.