Additional Material: Polynomial Basis

Differential Equations

Spring 2026

In the problem set this week, we have shown that $\{1, x, \cdots, x^n\}$ is linearly independent for all $n \in \mathbb{N}$. Now, let's think about how to consider that this "infinite set" is linearly independent.

Definition. Linearly Independence for Infinite Set.
An infinite set is linearly independent if all of its finite subsets are linearly independent.

(a) Show that $\{1, x, \cdots, x^n, \cdots\}$ is linearly independent.

Notice that we used the definition to verify all finite subsets, instead of having pairwise independence here, this definition has a reason.

(b) By an explicit construction, show that having pairwise linear independence is insufficient for linear independence.

We can also, similarly, expand the definition for spanning set to infinite dimensional space.

Definition. Spanning Set in Infinite Dimensional Vector Space.
A set is a spanning set if every element can be written as a finite linear combination with arbitrary small metric (must be defined for the space).

(c) Consider $C^\infty(\mathbb{R})$ be the set of smooth functions, equipped with the metric:

$$d(f_1, f_2) = \sup_{x \in \mathbb{R}} |f_1(x) - f_2(x)|.$$

Specifically, for the supremum, you may assume it as a minimal "upper bound" that can possible be $+\infty$. Is $\{1, x, \cdots, x^n\}$ a spanning set?

Eventually, in fact, polynomials can be more powerful than approximating smooth functions.

Definition. Bernstein Polynomial.
For any continuous function f over $C^0([0,1])$, we define its n-th order Bernstein polynomial as:

$$f_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Specifically, we have $\|f_n - f\| \to 0$.

Thus, we have the polynomials as a basis for $C^0([0,1])$ or any finite closed interval, which allows more possibilities later on.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) Recall that we have shown that we have shown that $\{1, x, x^2, \cdots, x^n\}$ is linearly independent for all $n \in \mathbb{N}$, hence for any finite subset of this set, there exists a maximum degree of the polynomial, say $N \in \mathbb{N}$, thus we have $\{1, x, \cdots, x^N\}$ is linearly independent, and therefore any subset of $\{1, x, \cdots, x^N\}$ must be linearly independent.

(b) Consider the functions:

$$f_1(x) = 1, \qquad f_2(x) = x, \qquad \text{and} \qquad f_3(x) = 1 + x,$$

we have:

$$\begin{aligned} \mathrm{W}[1, x] &= \det \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = 1, \\ \mathrm{W}[1, 1+x] &= \det \begin{pmatrix} 1 & 1+x \\ 0 & 1 \end{pmatrix} = 1, \qquad \text{and} \\ \mathrm{W}[1+x, x] &= \det \begin{pmatrix} 1+x & x \\ 1 & 1 \end{pmatrix} = 1 + x - x = 1, \end{aligned}$$

so they are pairwise linearly independence but:

$$\mathrm{W}[1, x, 1+x] = \det \begin{pmatrix} 1 & x & 1+x \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

(c) Yes, by the Taylor polynomial, we have for any smooth (analytic) function, we have:

$$f_N(x) = \sum_{i=0}^{N} \frac{1}{i!} f^{(i)}(x_0) \cdot x^i.$$

Specifically, by Taylor's Theorem, we have:

$$R_N(x) = f(x) - f_N(x),$$

where we have $|R_N(x)| \to 0$ as $N \to \infty$, and thus we have polynomials as a "dense" set for smooth polynomials.