Additional Material: Preview on Linear Algebra

Differential Equations

Spring 2026

As we start up this chapter, we can warm up on certain linear algebra topics so we can comprehend materials better. This problem reviews the basic concepts linear algebra concepts.

(a) Which of the following set of vectors are linearly independent in $\mathbb{R}$-vector space, what about $\mathbb{C}$-vector space? Justify your answer.

(i) $$\alpha = \{(1,1,0),(0,1,1),(1,0,1)\},$$

(ii) $$\beta = \{(0,1),(2,3),(4,5)\},$$

(iii) $$\gamma = \{1, \mathrm{i}\}.$$

(b) Let $A = \begin{pmatrix} 1+\mathrm{i} & -1+2\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{pmatrix}$ and $B = \begin{pmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{pmatrix}$, compute the following:

(i) $$A - 2B,$$

(ii) $$BA,$$

(iii) $$B^{-1}.$$

As we are familiarized with linear algebra concepts, we will work on an example on finding determinants:

(c) Find the determinant of A and B.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) For the first part, the $\mathbb{R}$ and $\mathbb{C}$-vector spaces should generally be the same:

(i) Consider the determinant of vertically concatenating the vectors that:

$$\det\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix} = 1+0+1-0-0-0 = 2 \neq 0,$$

hence it is $\boxed{\text{linearly independent}}$.

(ii) Here, the vector space is $\mathbb{R}^2$ or $\mathbb{C}^2$, which has dimension 2, but since there are three vectors, it is $\boxed{\text{not linearly independent}}$, or we have $(4,5) = 2\cdot(2,3) - 1\cdot(1,0)$.

(iii) This case is interesting, consider the $\mathbb{R}$-vector space, for any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have $\lambda_1 \cdot 1 + \lambda_2 \cdot \mathrm{i} = 0$ if and only if $\lambda_1 = \lambda_2 = 0$, so it is $\boxed{\text{linearly independent in } \mathbb{R}\text{-vector space}}$. Consider the $\mathbb{C}$-vector space, we have $1\cdot 1 + \mathrm{i}\cdot\mathrm{i} = 1-1 = 0$, so it is $\boxed{\text{not linearly independent in } \mathbb{C}\text{-vector space}}$.

(b) For the second part, we do the computation on the matrix operations:

(i) Consider $A - 2B$, we have:

$$\begin{aligned} A - 2B &= \begin{pmatrix}1+\mathrm{i} & -1+2\mathrm{i}\\ 3+2\mathrm{i} & 2-\mathrm{i}\end{pmatrix} - 2\begin{pmatrix}\mathrm{i} & 3\\ 2 & -2\mathrm{i}\end{pmatrix}\\ &= \begin{pmatrix}1+\mathrm{i}-2\mathrm{i} & -1+2\mathrm{i}-6\\ 3+2\mathrm{i}-4 & 2-\mathrm{i}+4\mathrm{i}\end{pmatrix} = \boxed{\begin{pmatrix}1-\mathrm{i} & -7+2\mathrm{i}\\ -1+2\mathrm{i} & 2+3\mathrm{i}\end{pmatrix}}. \end{aligned}$$

(ii) For BA, we do the matrix multiplication entry wise, that is:

$$BA = \begin{pmatrix}(1+\mathrm{i})\cdot\mathrm{i} + (-1+2\mathrm{i})\cdot 2 & \mathrm{i}(-1+2\mathrm{i}) + 3(2-\mathrm{i})\\ (3+2\mathrm{i})\cdot\mathrm{i} + (2-\mathrm{i})\cdot 2 & (3+2\mathrm{i})\cdot 3 + (2-\mathrm{i})\cdot(-2\mathrm{i})\end{pmatrix} = \boxed{\begin{pmatrix}8+7\mathrm{i} & 4-4\mathrm{i}\\ 6-4\mathrm{i} & -4\end{pmatrix}}.$$

(iii) To find the inverse, we can use the formula that:

$$B^{-1} = \frac{1}{\det B}\begin{pmatrix}-2\mathrm{i} & -3\\ -2 & \mathrm{i}\end{pmatrix} = \frac{1}{-4}\begin{pmatrix}-2\mathrm{i} & -3\\ -2 & \mathrm{i}\end{pmatrix} = \boxed{\begin{pmatrix}\mathrm{i}/2 & 3/4\\ 1/2 & -\mathrm{i}/4\end{pmatrix}}.$$

(c) The determinant of A can be computed as follows:

$$\det\begin{pmatrix}1+\mathrm{i} & -1+2\mathrm{i}\\ 3+2\mathrm{i} & 2-\mathrm{i}\end{pmatrix} = (1+\mathrm{i})(2-\mathrm{i}) - (3+2\mathrm{i})(-1+2\mathrm{i}) = 3+\mathrm{i} - (-7+4\mathrm{i}) = \boxed{10-3\mathrm{i}}.$$

Similarly, for B, we can also compute the determinant quite directly:

$$\det\begin{pmatrix}\mathrm{i} & 3\\ 2 & -2\mathrm{i}\end{pmatrix} = \mathrm{i}\cdot(-2\mathrm{i}) - 2\times 3 = 2 - 6 = \boxed{-4}.$$