PILOT

Additional Material: Matrix Exponential

Differential Equations

Spring 2026

This set discusses with a new concept called **Matrix exponential**, which is not covered in this semester, but very interesting to investigate on.

Definition. (Matrix Exponential). Similar to how we defined the exponential function analytically, the exponential function is also defined for matrices, let A be a square matrix, we define:

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i.$$

The matrix exponential will be interesting in the following classes of matrices.

Definition. (Nilpotent Operator). Let M be a square matrix, M is *nilpotent* if $M^k = 0$ for some positive integer k.

(a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

(b) Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
Hint: Use the differences of squares for matrices.

(c) If all the entries in N are rational, show that $\exp(N)$ has rational entries.

Similarly, we can also define such matrices with another class of interesting matrices.

Definition. (Rotational Matrix). Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle θ (counter-clockwise), then:

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(d) Show that $M^{\intercal} = M^{-1}$.

(e) Let $\theta = 2\pi/k$ be fixed, where k is an integer. Find the least positive integer n such that $M^n = \mathrm{Id}_2$. Here, n is called the *order* of M.
Hint: Consider the rotational matrix geometrically, rather than arithmetically.

(f) Let $\theta = \pi/2$, calculate the matrix exponential $\exp(M)$.
Hint: Consider the *order* of M and the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$.

The solutions to this additional problem is on the next page...

Solutions to the Additional Problem:

(a) *proof of N is nilpotent.* Here, we want to do the matrix multiplication:

$$N^2 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N^3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we have shown that $N^3 = 0$, or the zero matrix, hence N is nilpotent. □

Then, we want to calculate the matrix exponential, that is:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}.$$

(b) *Proof.* Here, we recall the differences of squares still works when commutativity for multiplications fails, hence the we can still use it for matrix multiplication, namely, for all $m \in \mathbb{Z}^+$:

$$(\mathrm{Id}_n + N) \cdot (\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) = \mathrm{Id}_n - N^{2^{m+1}}$$

Since N is *nilpotent*, this implies that we have some k such that $N^\ell = 0$ for all $\ell \geq k$. Meanwhile, note that $2^\ell \geq \ell$ for all positive integer ℓ. (This can be proven by induction.) Therefore, we select $m + 1 \geq k$ so that $N^{2m+1} = 0$, and we have:

$$(\mathrm{Id}_n + N) \cdot \left[(\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) \right] = \mathrm{Id}_n,$$

thus $\mathrm{Id}_n + N$ is invertible. □

(c) *Proof.* By the definition that N is nilpotent, we know that $N^m = 0$ for some finite positive integer m, hence, we can make the (countable) infinite sum into a finite sum:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \sum_{k=0}^{m} \frac{1}{k!} N^k,$$

thus all the entries are sum and non-zero divisions of rational numbers, while rational numbers are closed under addition and non-zero divisions, hence, all entries of $\exp(N)$ is rational. □

Note that the elements of all n-by-n matrices can be considered as a *ring*, while *nilpotent* can be defined more generally for *rings*. We invite capable readers to investigate more properties of *nilpotent* elements of *rings* in the discipline of *Modern Algebra*.

(d) *Proof.* Here, we recall the method of inverting a matrix:

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} \cos\theta & -(-\sin\theta) \\ -\sin\theta & \cos\theta \end{pmatrix} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = M^\intercal.$$

□

(e) Look, we want to analyze this geometrically, if $\theta = 2\pi/k$, then that implies that M is a counter-clockwise rotation of $2\pi/k$, and since a full revolution is 2π, this implies a rotation of k times will make restore to the original vector, *i.e.*, $M^k = \mathrm{Id}_2$. Moreover, for any positive integer less than k, we cannot rotate back to 2π, which implies that the order of M is $\boxed{k}$.

(f) Here, we construct the matrix exponential, note that the order of M is 4, we have:

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$

Here, we want to consider each entry respectively, since each entry is finite and since M has order 4, the absolute value of the sum of the entries must be finite, so each entry converges *absolutely*, hence we are free to change the order of the sum, so we have:

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{(4k+1)!} M + \sum_{k=0}^{\infty} \frac{1}{(4k+2)!} M^2 + \sum_{k=0}^{\infty} \frac{1}{(4k+3)!} M^3 + \sum_{k=0}^{\infty} \frac{1}{(4k)!} \operatorname{Id}.$$

For the 4 sums of factorials, we note that the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$ at 0 evaluated at $x = 1$ are, respectively:

$$\begin{array}{lllllllll} e^1 & = & +\frac{1}{0!} & +\frac{1}{1!} & +\frac{1}{2!} & +\frac{1}{3!} & +\frac{1}{4!} & +\frac{1}{5!} & +\cdots \\ e^{-1} & = & +\frac{1}{0!} & -\frac{1}{1!} & +\frac{1}{2!} & -\frac{1}{3!} & +\frac{1}{4!} & -\frac{1}{5!} & +\cdots \\ \sin 1 & = & & +\frac{1}{1!} & & -\frac{1}{3!} & & +\frac{1}{5!} & -\cdots \\ \cos 1 & = & +\frac{1}{0!} & & -\frac{1}{2!} & & +\frac{1}{4!} & & -\cdots \end{array}$$

Since the first series converges, we know that the later three series converges *absolutely*, so we are free to move around terms.

From the expressions, by columns, we can observe that:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} = \frac{e^1 - e^{-1}}{4} + \frac{\sin 1}{2}, \qquad \sum_{k=0}^{\infty} \frac{1}{(4k+2)!} = \frac{e^1 + e^{-1}}{4} - \frac{\cos 1}{2},$$

$$\sum_{k=0}^{\infty} \frac{1}{(4k+3)!} = \frac{e^1 - e^{-1}}{4} - \frac{\sin 1}{2}, \qquad \sum_{k=0}^{\infty} \frac{1}{(4k)!} = \frac{e^1 + e^{-1}}{4} + \frac{\sin 1}{2}.$$

Now, we shall also evaluate the matrices generated by M, that is:

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$M^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, considering the four entries $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have:

$$a = -\frac{e + 1/e}{4} + \frac{\cos 1}{2} + \frac{e + 1/e}{4} + \frac{\sin 1}{2} = \frac{\cos 1 + \sin 1}{2},$$

$$b = -\frac{e - 1/e}{4} - \frac{\sin 1}{2} + \frac{e - 1/e}{4} - \frac{\sin 1}{2} = -2\sin 1,$$

$$c = \frac{e - 1/e}{4} + \frac{\sin 1}{2} - \frac{e - 1/e}{4} + \frac{\sin 1}{2} = 2\sin 1,$$

$$d = -\frac{e + 1/e}{4} + \frac{\cos 1}{2} + \frac{e + 1/e}{4} + \frac{\sin 1}{2} = \frac{\cos 1 + \sin 1}{2}.$$

Therefore, the matrix exponential is:

$$\exp(M) = \boxed{\begin{pmatrix} \dfrac{\cos 1 + \sin 1}{2} & -2\sin 1 \\ 2\sin 1 & \dfrac{\cos 1 + \sin 1}{2} \end{pmatrix}}.$$

In particular, mathematicians has considered the *rotation* and *flipping* of regular polygons as the *dihedral groups*, where symmetries and combinatorics play an important role. Please think of ways you may "manipulate" a polygon such that the polygon looks the same.